COMP 761: Lecture 10 – Linear Algebra I

David Rolnick

September 25, 2020

Course Announcements

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- Vincent's optional list of practice problems available on Slack

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- Vincent's optional list of practice problems available on Slack
- Monday class attendance optional if observing Yom Kippur

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- One can also work with vectors where the v_k are complex numbers, rational numbers, etc, but we won't need that
- Order matters, so e.g. $[2, 1, 1] \neq [1, 2, 1]$
- When working with real vectors, we say real numbers are *scalars* to distinguish from vectors. So $[2, 2]$ is a vector, 2 is a scalar.

Basic properties

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- Vector addition and subtraction are defined as performing those operations component-wise:

$$[v_1, \ldots, v_n] + [w_1, \ldots, w_n] = [v_1 + w_1, \ldots, v_n + w_n]$$
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- Multiplication or division of a vector by a scalar also works component-wise:

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$$5v - 2w = [5, 0] - [0, 2] = [5, -2].$$

Graphical representations

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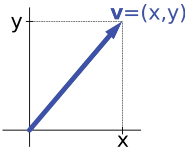
- Often represented by points in Cartesian coordinates:



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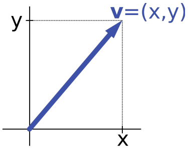
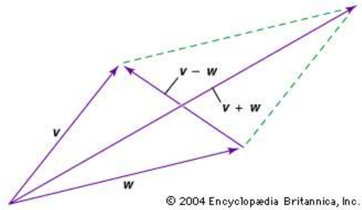


- Or by arrows with a fixed length and direction:



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- For $p \geq 1$, the ℓ^p*-norm* (or just *p-norm*) of a vector $x = (x_1, \ldots, x_n)$ is the scalar value:

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- Often, $||x||_2$ is just written $||x||$.

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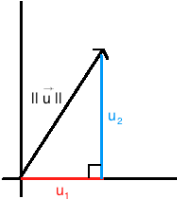
- Geometrically, the ℓ^2-norm of a vector is equal to its length:



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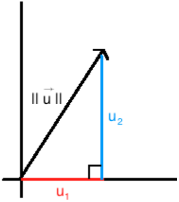

- By the Pythagorean Theorem:

$$\text{Length}^2 = u_1^2 + u_2^2 = ||u||_2^2.$$

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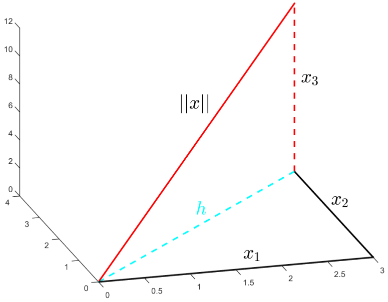
- Why is this true in higher dimensions?



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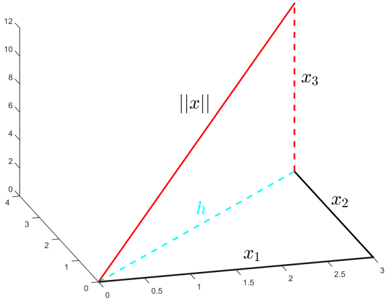

- In 3D, we can apply the Pythagorean Theorem twice:

$$x_1^2 + x_2^2 = h^2$$
$$h^2 + x_3^2 = \text{Length}^2$$
$$\text{so Length}^2 = x_1^2 + x_2^2 + x_3^2 = ||x||_2^2.$$

The ℓ^∞ norm

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- Let's look at some different norms of the vector $x = [1, -10, 8, 0]$:

$$||x||_1 = 19, \quad ||x||_2 \approx 12.85, \quad ||x||_5 \approx 10.58, \ \ldots, \quad ||x||_{20} \approx 10.01.$$

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- That is because for p large,

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- Formally, we define the ℓ^∞-norm of x to be $\max_k |x_k|$.

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- Technically, this is not a mathematical "norm" (because $||cx||_0 \neq |c| \cdot ||x||_0$), but still very useful.

Dot products

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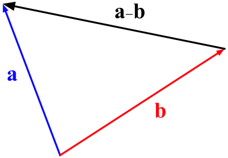

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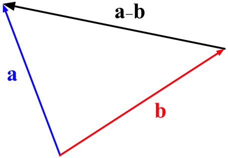

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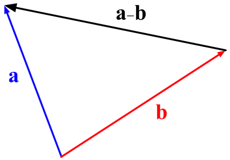

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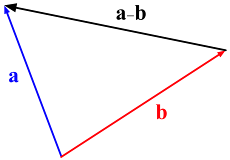

- $a, b, a-b$ form a triangle.
- We know the lengths of the sides: $||a||_2, ||b||_2, ||a-b||_2$.
- There is a geometric result called the Law of Cosines (generalizing the Pythagorean Theorem) between the sides a, b, c of any triangle:

$$\text{len}(a)^2 + \text{len}(b)^2 = \text{len}(c)^2 + 2\text{len}(a)\text{len}(b)\cos\theta,$$

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- We can substitute in our lengths $||a||_2, ||b||_2, ||a-b||_2$:

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- We can expand this:

$$\left(a_1^2 + \cdots + a_n^2\right) + \left(b_1^2 + \cdots + b_n^2\right)$$
$$= \left((a_1 - b_1)^2 + \cdots + (a_n - b_n)^2\right) - 2||a||_2||b||_2 \cos\theta$$
$$= \left(a_1^2 - 2a_1b_1 + b_1^2\right) + \cdots + \left(a_n^2 - 2a_nb_n + b_n^2\right) + 2||a||_2||b||_2 \cos\theta$$

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- We can expand this:

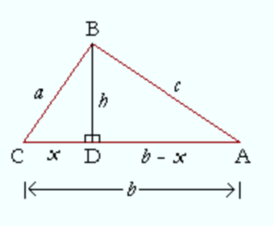
$$\begin{aligned}
&\left(a_1^2 + \cdots + a_n^2\right) + \left(b_1^2 + \cdots + b_n^2\right) \\
&= \left((a_1 - b_1)^2 + \cdots + (a_n - b_n)^2\right) - 2||a||_2||b||_2 \cos\theta \\
&= \left(a_1^2 - 2a_1b_1 + b_1^2\right) + \cdots + \left(a_n^2 - 2a_nb_n + b_n^2\right) + 2||a||_2||b||_2 \cos\theta
\end{aligned}$$

- Therefore:

$$2a_1b_1 + \cdots + 2a_nb_n = 2||a||_2||b||_2 \cos\theta,$$

and therefore $a \cdot b = ||a||_2||b||_2 \cos\theta$.

So why is the Law of Cosines true?



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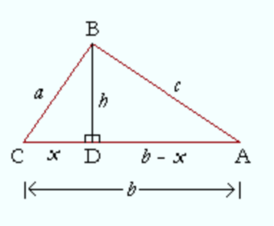

- We can use the Pythagorean Theorem again:

$$
\begin{aligned}
a^2 &= x^2 + h^2 \\
&= x^2 + (c^2 - (b - x)^2) \\
&= x^2 + (c^2 - b^2 + 2bx - x^2) \\
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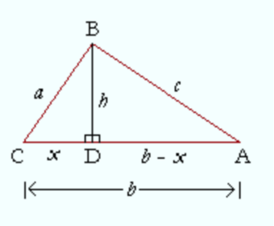

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- (Note: A slightly different proof is needed for obtuse triangles.)

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- If a and b are perpendicular, then $\theta = 90^\circ$, so $\cos\theta = 0$ and therefore

$$a \cdot b = 0.$$

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- This is often represented:

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- Vector addition and subtraction are defined as performing those operations component-wise:

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- Multiplication (or division) of a vector by a scalar also works component-wise:

$$c \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$$

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- More generally, a permutation matrix A has the property that Ax is a permutation of the entries of x.

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- Multiplying matrix $A \in \mathbb{R}^{m \times n}$ by matrix $B \in \mathbb{R}^{n \times p}$ is like multiplying A by each column vector of B and combining the resulting columns.

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- The resulting product AB is a matrix in $\mathbb{R}^{m \times p}$.

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- Proof for $A \in \mathbb{R}^{1\times m}$, $B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{n\times 1}$.

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- So

$$\begin{aligned} (AB)C &= (a_1 b_{11} + a_2 b_{21} + \cdots + a_m b_{m1})\, c_1 + \cdots \\ &\quad (a_1 b_{1n} + a_2 b_{2n} + \cdots + a_m b_{mn})\, c_n \\ &= \sum_{1\leq i\leq m, 1\leq j\leq n} a_i b_{ij} c_j. \end{aligned}$$

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- Simple property: $(A^T)^T = A$.
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- $(CA)^T = A^T C^T = AC^T = C^T$, so $CA = C$.

Next time!

Linear Algebra II