COMP 761: Lecture 11 – Linear Algebra II

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Problem

An *eigenvector* of a matrix $A \in \mathbb{R}^{n \times n}$ is a nonzero vector $x \in \mathbb{R}^n$ such that $Ax = \lambda x$, where the value λ is called an *eigenvalue*. The trace of a square matrix is the sum of the values along the diagonal. Prove that the trace of a matrix equals the sum of the eigenvalues.

(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)

Course Announcements

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- Problem Set 2 available on Slack and MyCourses

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- Office hours today right after class

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- There can be at most n.

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- Are the row and column rank always equal? Why?

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- Similarly (using A^T), column rank $\leq$ row rank, so they are equal.

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- All the columns are linearly independent, so rank = n (*full-rank*).

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- Therefore, for some B, we have $AB = I$.

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We write A^{-1} for the inverse of A.

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- If A is invertible, the (unique) solution is $v = A^{-1}b$.

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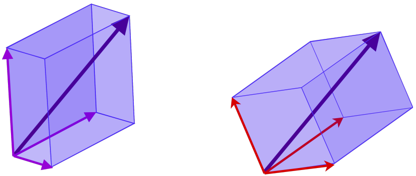
- (ii) means the norm of every column is 1.

Orthogonal matrices

- An *orthogonal matrix* A is one where $A^T A = I$, so $A^T = A^{-1}$.
- For example: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}.$
- Equivalent to saying that (i) every two different columns of A have dot product 0, (ii) the dot product of any column with itself is 1.
- (i) means any two columns are perpendicular (aka *orthogonal*).
- (ii) means the norm of every column is 1.
- These two conditions mean the columns of an orthogonal matrix form a *basis* (essentially a different set of coordinate axes):



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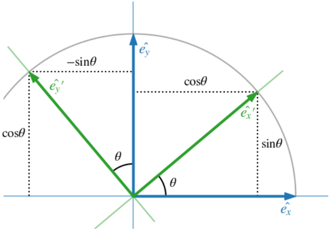

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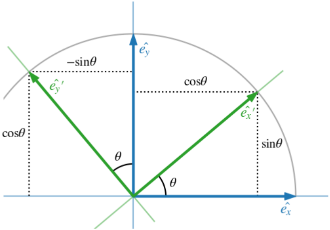

- For example, to rotate $[3,4]$ by 45°, we compute:

$$\begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 \\ 7\sqrt{2}/2 \end{bmatrix}$$

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- $n \times n$: If $A \in \mathbb{R}^{n \times n}$, let A_k be obtained from A by deleting row 1 and column k. Then,

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- Each one of the products, e.g. $a_{13}a_{21}a_{32}$, will have one entry in each column, and also one entry in each row.
- The sign of the product will be the *sign of the permutation* (can look up if you are curious).

Determinant facts

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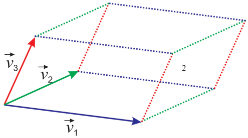
1. The determinant is $\pm$ the volume of the parallelepiped formed by the columns of the matrix.



$$V = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

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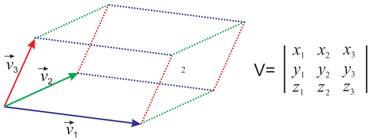

2. From this, we can conclude that $\det(A) = 0$ unless the columns of A span the whole space $\mathbb{R}^n$. That is, all of these are equivalent for $A \in \mathbb{R}^{n \times n}$:

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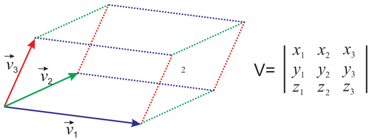

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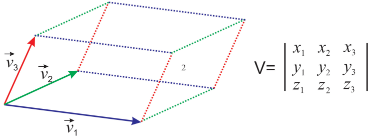

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- In practice, we often scale x so that $||x|| = 1$.

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- The characteristic polynomial's roots = the eigenvalues (may be complex).

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- But by Vieta's formulas, this coefficient is just:

$$\sum_{k=1}^{n} a_{kk} = \mathrm{tr}(A).$$

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- By Vieta's formulas, the product of the roots equals $\det(A)$.

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Next time!

Graph Theory II