COMP 761: Lecture 2 – Proofs

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September 4, 2020

Problem

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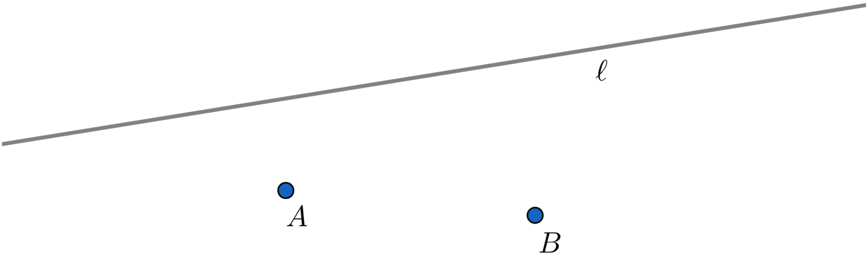

Course Announcements

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- TA: Vincent Luczkow

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- Problem Set 1 will be released Tuesday, due Friday Sept 18

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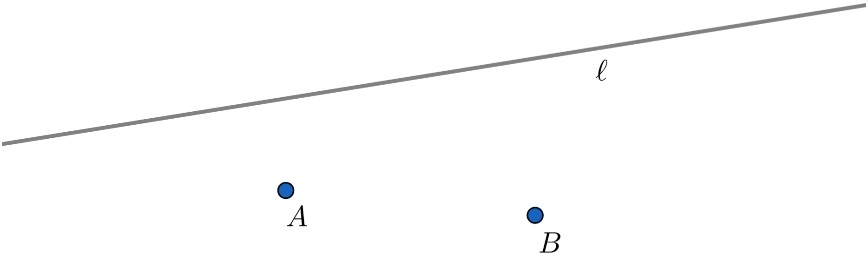
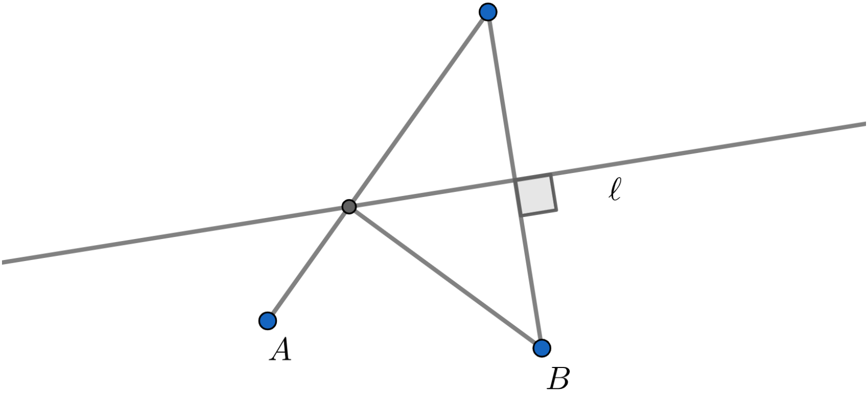


Answer



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You can think of an *answer* as something short ($x = 2$) but a proof as the full *solution* (this is why $x = 2$)

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- In class, sometimes full proofs, sometimes just intuitions for why true, if proof is hard
- Useful to be able to understand a proof might look like, even if we don't always dive into it

Proof-writing tips

1. Indicate how you are making each conclusion

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This is also good to make sure you aren't assuming what you are trying to prove!

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- Bad: *Then, the algorithm runs in time $O(n + \text{the right-hand side})$.*

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- Bad: $x = y^2, y = z^2, \Rightarrow x = z^4$.

Also, remember to break into paragraphs, otherwise it can get very hard to read.

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- Good: *Squaring both sides of the equation, we obtain* $(x-1)^2 = \sin(z - \sqrt{3}\omega)$.
- Bad: *What happens if you square both sides of the equation? You get* $(x-1)^2 = \sin(z - \sqrt{3}\omega)$*!*

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- Good: *In figure 1, point P is the intersection of segment AB with line L.*

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- Good: *We will prove the result by induction, dividing into 3 cases according to whether a is positive, negative, or* 0. *We begin by proving the following Claim.*

  Claim: b is even.
  Proof of Claim: ....

  Having proven the claim, we proceed to our three cases:

  Case 1: $a > 0$
  Proof in Case 1: ....

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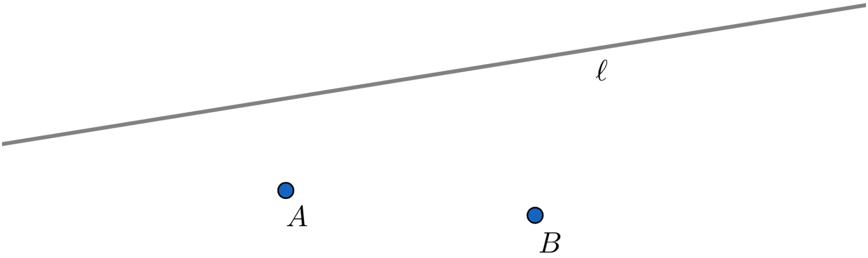

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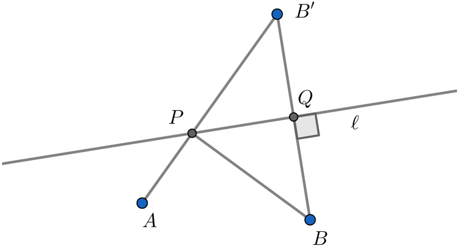
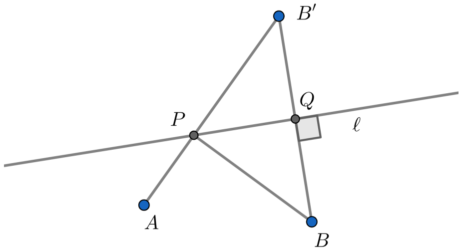


To minimize the distance traveled, it is optimal to travel from A to some point P along ℓ, and then from P to B. We therefore must find the point P on ℓ such that $AP + BP$ is minimized.

Let B' be the reflection of point B across line ℓ (that is, BB' is perpendicular to ℓ and B and B' are equidistant from ℓ), as shown in the Figure. Let Q be the intersection of BB' with ℓ.

Example proof



Claim: for any point P on ℓ, the distance $B'P$ equals the distance BP.
Proof of Claim: By the Pythagorean Theorem on right triangle PQB:

$$BP^2 = PQ^2 + BQ^2.$$

Similarly, from the Pythagorean Theorem on right triangle PQB', we obtain:

$$B'P^2 = PQ^2 + B'Q^2.$$

Combining these two equations and using the fact that $B'Q = BQ$, we conclude that $BP = B'P$, proving the claim.

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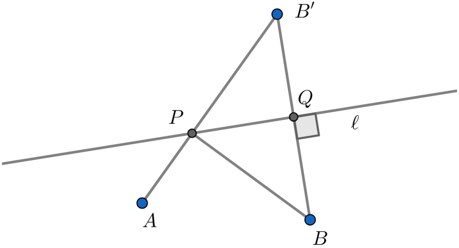

Now, since $B'P = BP$ for any P, we know that minimizing $AP + BP$ is equivalent to minimizing $AP + B'P$. Since a straight line is the shortest distance between two points, the optimum is attained where P is the intersection of AB' with ℓ.

Thus, our algorithm is to reflect B across ℓ to B' and to take the intersection of AB' with ℓ.

■

Problem (Monty Hall).

You are in a game show where you can win a prize. There are three doors. Behind two of them are goats. Behind the third is a car. (Assume for this problem that you want a car more than a goat.) You pick a door at random. The host (who knows what is where) then opens one of the other doors, revealing a goat. You can either stick with your door, or switch to the other one. Which is better?

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Since Case 1 occurs with 1/3 probability and Case 2 with 2/3 probability, 2/3 of the time it will be optimal to switch. Therefore, switching is the best strategy.
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4. Think about what info you haven't used
   - What conditions are necessary or else it wouldn't work?
   - Good sign you'll need to use them!

Problem.

Out of any 1000 integers, prove that some subset of them sum to a multiple of 1000.

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- But 1000 is a large number
- Let's try 10 numbers
- Maybe $1, 2, 3, 4, 5, 6, 6, 8, 9, 10$...whoops that is easy
- How can that be harder?

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- $1 + 2 = 3$
- $1 + 2 + 3 = 6$
- $1 + 2 + 3 + 5 = 11$

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- $1 = 1$
- $1 + 2 = 3$
- $1 + 2 + 3 = 6$
- $1 + 2 + 3 + 5 = 11$
- $1 + 2 + 3 + 5 + 6 = 17$

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- $1 + 2 + 3 + 5 = 11$
- $1 + 2 + 3 + 5 + 6 = 17$
- $1 + 2 + 3 + 5 + 6 + 7 = 24$

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- $1 + 2 + 3 + 5 + 6 + 7 + 8 + 1 = 43$

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- $8 + 1 + 1 = 10$

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Next time!

Proof techniques: Induction