COMP 761: Lecture 3 – Induction

David Rolnick

September 9, 2020

Problem

Prove the formula for *triangular numbers*:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

How about the analogous formula for squares?

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Course Announcements

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- Problem Set 1 is out, due Friday Sept 18 at 11:59 pm.

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- Formulas for sequences of numbers
- Proving what happens in a recursive algorithm
- When something can be reduced to a smaller problem (e.g. a “divide and conquer” approach)

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showing the claim holds for $n = k + 1$. This completes the induction. ■

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- With some algebra, can show that:
$$\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}.$$

Problem

The Fibonacci numbers $(0, 1, 1, 2, 3, 5, 8, 13, \ldots)$ are defined by:

$$
\begin{aligned}
F_0 &= 0, \\
F_1 &= 1, \\
F_n &= F_{n-1} + F_{n-2} \text{ for } n \geq 2.
\end{aligned}
$$

Prove the following:

$$F_n = \frac{1}{\sqrt{5}} \left(\phi^n - \bar{\phi}^n \right),$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the *golden ratio* and $\bar{\phi} = \frac{1-\sqrt{5}}{2} = 1 - \phi$.

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- Let's assume:

$$F_{k-1} = \frac{1}{\sqrt{5}} \left(\phi^{k-1} - \bar{\phi}^{k-1} \right),$$
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- Useful fact: ϕ and $\bar{\phi}$ are the solutions to the equation $x^2 - x - 1 = 0$ (by the Quadratic Formula), that is:

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By our inductive hypothesis, the RHS equals $F_k + F_{k-1}$, which equals F_{k+1} by the Fibonacci relation. We conclude that:

$$F_{k+1} = \frac{1}{\sqrt{5}}\left(\phi^{k+1} - \bar{\phi}^{k+1}\right).$$

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$$F_0 = 0 = \frac{1}{\sqrt{5}}\left(\phi^0 - \bar{\phi}^0\right)$$
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This completes our proof. ■

General case of this logic

Problem. Given a linear recurrence relation:

$$A_0 = a_0,\ A_1 = a_1,\ \cdots\ ,\ A_{k-1} = a_{k-1},$$
$$A_n = w_1 A_{n-1} + w_2 A_{n-2} + \cdots + w_k A_{n-k} \text{ for } n \geq k,$$

can we express A_n with a closed form equation?

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can we express A_n with a closed form equation?

Answer. In general, if $\phi_1, \phi_2, \ldots, \phi_k$ are the solutions to the polynomial equation:

$$x^k - w_1 x^{k-1} - w_2 x^{k-2} - \cdots - w_k = 0,$$

then we have

$$A_n = c_1 \phi_1^n + c_2 \phi_2^n + \cdots + c_k \phi_k^n,$$

where the constants c_1 are determined by initial conditions $a_0, a_1, \ldots, a_{k-1}$.

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- Try random moves, Alice takes 1, so 3 left.
- Bob takes 1, so 2 left.
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- Nope, just 1 stone left, so Alice wins regardless.

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- What if $n = 5$?
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- What happens now?

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- Same as before, whatever Bob does, Alice wins.

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- Maybe multiples of 3 are bad for Alice.

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In this case, if Alice removes 1 stone, Bob gets a pile of k stones, which by our inductive hypothesis we know is a winning position (since k is not divisible by 3). If Alice removes 2 stones, Bob gets a pile of $k - 1$ stones, which is also a winning position. Therefore, Alice loses.

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Thus, from our casework we confirm that $k + 1$ is a winning position for Alice if and only if it is not a multiple of 3.

Finally, we confirm that our base cases hold. Since we required assumptions on k and $k - 1$, we must check two base cases: $n = 1$ and $n = 2$. Neither is divisible by 3 and it is clear that both of these are indeed winning positions for the starting player.

■

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Some summation formulas that can be proved with induction can also be proved using *telescoping sums*, e.g.

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Next time!

Proof techniques: Contradiction