COMP 761: Lecture 4 – Contradiction

David Rolnick

September 11, 2020

Problem

An infinite set of numbers is said to be *countable* if you can list them all. For example, the positive integers are countable:

$$1, 2, 3, 4, \ldots$$

So are all the integers:

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

Are the real numbers countable? Why or why not?

Course Announcements

Problem sets - what math can you use?

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- Also, office hours (mine are Mon 5pm and Fri 10am, Vincent's are Thu 10:30)

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- Conclude that the thing you want is actually true.

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- Contradiction!

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This is a contradiction, and we conclude that there are infinitely many primes. ■

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- Proving that something doesn't exist (e.g. a largest prime number).
- Proving that some property always holds.
- A lot of other situations...

Preamble: Countability

An infinite set of numbers is said to be *countable* if you can list them all. For example, the set $\mathbb{Z}^+$ of positive integers is countable:

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So is the set $\mathbb{Z}$ of all integers:

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Warmup: Are the positive even integers countable?

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- Note that a countable set can contain another countable set.

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- But what about $1/3$?

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- Better way? Or does it not work?

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- Here is a way to visualize them:

$$\begin{array}{ccccccc} 1/1 & 1/2 & 1/3 & 1/4 & 1/5 & \cdots \\ 2/1 & 2/2 & 2/3 & 2/4 & 2/5 & \cdots \\ 3/1 & 3/2 & 3/3 & 3/4 & 3/5 & \cdots \\ 4/1 & 4/2 & 4/3 & 4/4 & 4/5 & \cdots \\ 5/1 & 5/2 & 5/3 & 5/4 & 5/5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

- Is this helpful?

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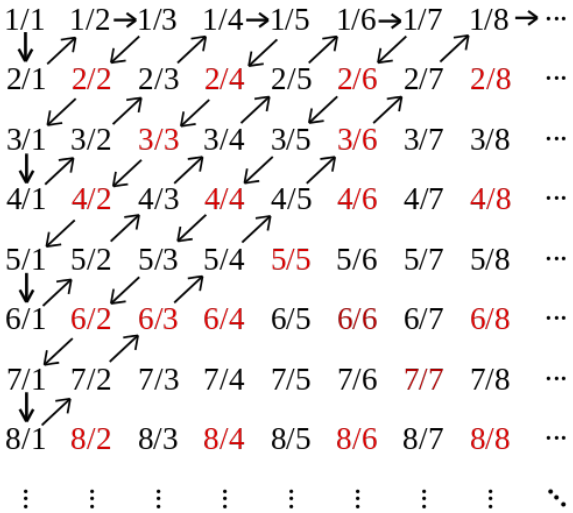

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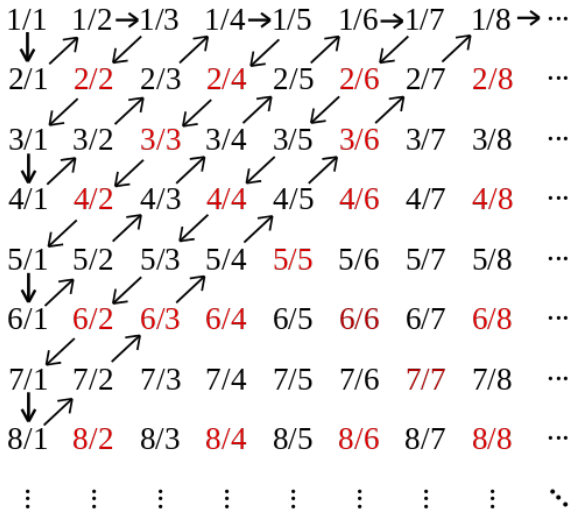

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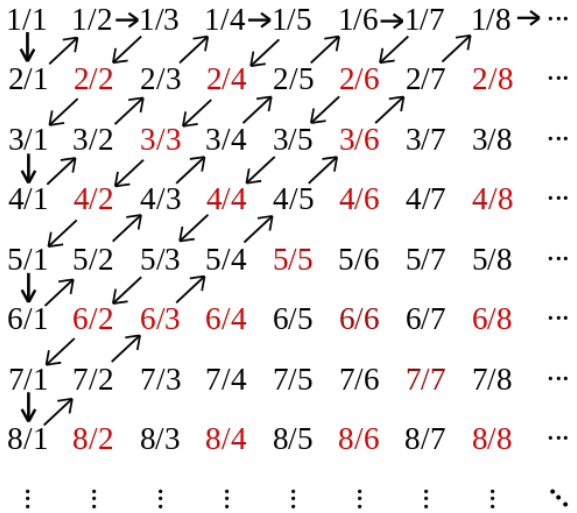

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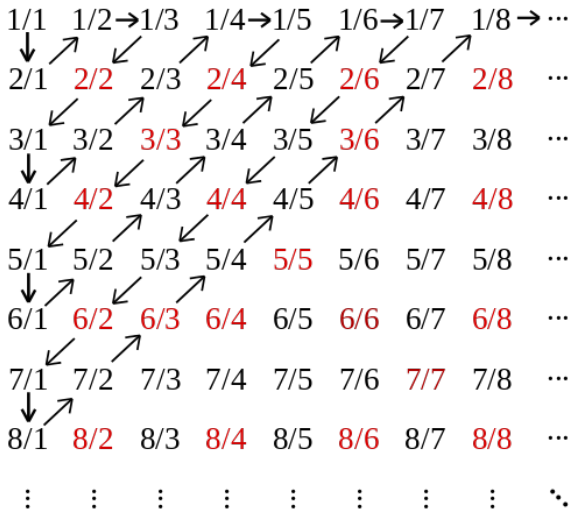

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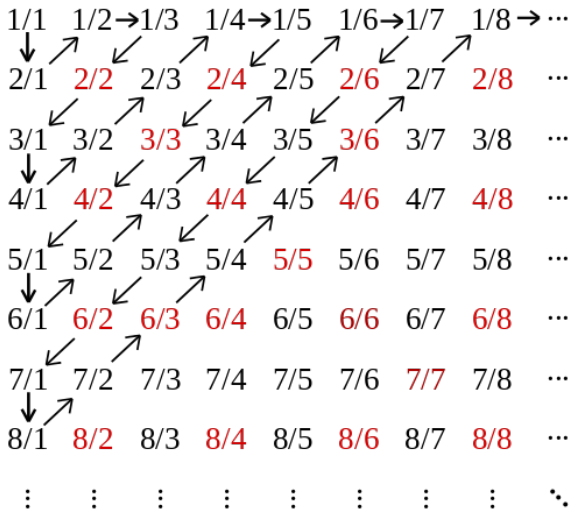

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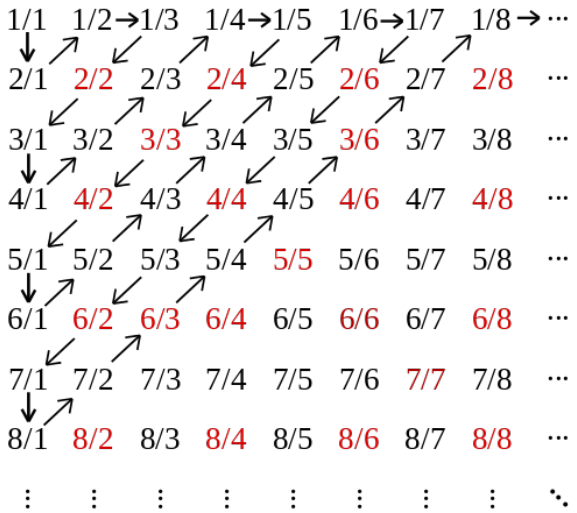

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- We can alternate between positive and negative to get both
- So $\mathbb{Q}$ is countable! (not a formal proof)

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- Well, this class is on contradiction...
- Let's actually prove something stronger, that the set $[0, 1]$ of real numbers between 0 and 1 is uncountable.

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- Since we know real numbers have infinite representations, let's write that out:

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\begin{aligned}
x_1 &= 0.x_{11}x_{12}x_{13}x_{14}\cdots \\
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- How can we define y?

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- So it has to be a number that isn't in our list - contradiction.

Another problem

There are 5 points inside an equilateral triangle of side length 1. Prove that some two of them are at distance at most $1/2$ from each other.

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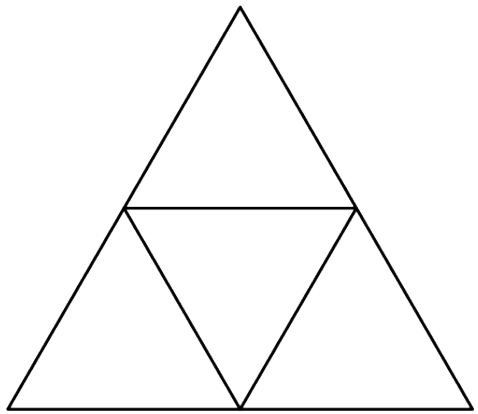
Suppose towards contradiction that this claim is not true. Then, every two points are at distance more than $1/2$.

Consider subdividing the triangle into 4 smaller equilateral triangles, each with side length $1/2$, as shown in the figure. By the Pigeonhole Principle, at least 2 out of the 5 points must lie in the same subtriangle. However, for each subtriangle, the maximum distance between points is $1/2$ – giving us a contradiction. We conclude that the result holds. ■



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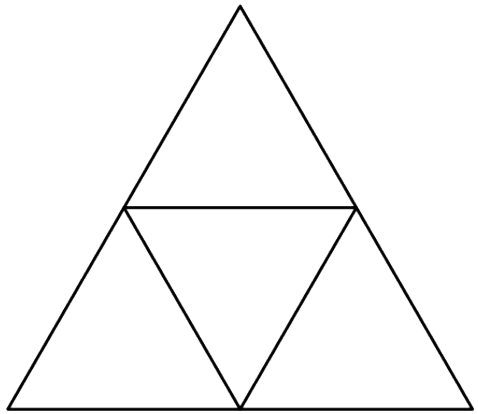
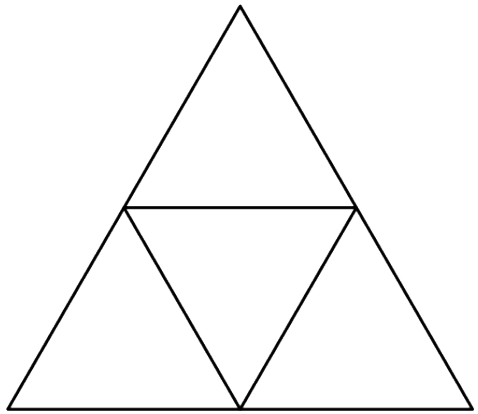


What's bad about this proof?

That proof is correct, but there is no need for the contradiction! We can rewrite it:

Consider subdividing the triangle into 4 smaller equilateral triangles, each with side length $1/2$, as shown in the figure. By the Pigeonhole Principle, at least 2 out of the 5 points must lie in the same subtriangle. Since, for each subtriangle, the maximum distance between points is $1/2$, we are done. ■



Try not to write proofs where you assume the claim is false, but never use that fact. (It's not wrong, but it's unnecessary and can be confusing to the reader.)

Preamble: Turing machines

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- A *Turing machine* is a machine that transitions between a finite number of states based upon (i) the state where it currently is, (ii) the part of memory it is reading from at the moment.
- A *state* can include directions to write to memory or move to a different part of memory.

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- When we talk about Turing machines, it's about things a computer *can* compute, not how fast or efficiently. Turing machines are a mathematical abstraction – they are used in working out whether something can be computed or not, not the best way of doing it.
- **All you need to know:** Given an input, a Turing machine can do one of three things: **accept** (output yes), **reject** (output no), or **loop** forever.

Problem

Prove there is no Turing machine that can do the following: Given M and w, where M is a Turing machine, **accept** (M, w) if M accepts w and **reject** (M, w) if M rejects w or if M loops forever.

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- Since we have the magic simulator H, we know how any Turing machine acts for input w.

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- A would do **whatever its input B does when applied to itself**.
- So B has to do the opposite of what it does! Contradiction.

The Halting Problem

As a corollary, we can prove the famous Halting Problem, which says it's impossible to determine in general whether a Turing machine will loop forever or "halt" (i.e., accept or reject).

Theorem. There is no Turing machine that, given input (M, w), outputs **accept** when M halts on input w and **reject** when M loops forever.

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- We just reject it.
- Since we knew the previous problem *couldn't be solved*, we get a contradiction.

Next time!

Other proof techniques
(invariants, monovariants, inequalities)