COMP 761: Lecture 6 – Polynomials

David Rolnick

September 16, 2020

Problem

What is the minimum possible value of $x^{100} + \frac{3}{x^{100}}$?

(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)

Course Announcements

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- Office hours: Vincent on Thursday at 10:30 am, David on Friday at 10 am

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$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where $a_0, a_1, \ldots, a_n$ are constants called *coefficients*.

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- Degree 1 (*linear*): e.g. $4x + 4$.
- Degree 0 (*constant*): e.g. 17.

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- Example: $p(x) = 2x^3 + 2x + 1$, $q(x) = x^2 + x$,

$$2x^3 + 2x + 1 = (x^2 + x)(2x - 2) + (4x + 1),$$

with $4x + 1$ the remainder (with degree 1, which is strictly less than the degree of $q(x)$).

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- $x - r$ is degree 1, so $s(x)$ is degree 0 – that is, it's a constant s.
- What happens if we set $x = r$?

$$p(r) = q(r)(r - r) + s = s,$$

  so $s = 0$ if and only if $p(r) = 0$.

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- Example: $(x-3)^2 = x^2 - 6x + 9$,

$$b^2 - 4ac = 36 - 36 = 0.$$

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- And $i^4 = (i^2)^2 = (-1)^2 = 1$.

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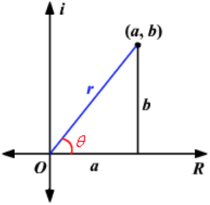
- We often consider complex numbers $a + bi$ in the *complex plane*. That just means taking them as points in 2D space, with x-coordinate a and y-coordinate b.



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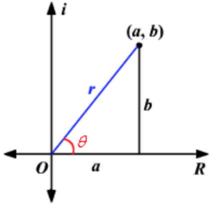

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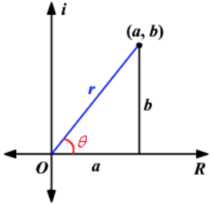

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- An awesome thing about multiplying complex numbers:

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- Can prove this using trigonometric formulas, but the *real reason* will have to wait a few lectures until Taylor series.

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- This makes sense since we know $(4i)^2 = 16i^2 = -16$.

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- We can take $\theta = 45^\circ$, so
$$x = 2(\cos(45^\circ) + i\sin(45^\circ)) = \sqrt{2} + \sqrt{2}i.$$
- Another option: $\theta = (90^\circ + 360^\circ)/2 = 225^\circ$, so
$$x = 2(\cos(225^\circ) + i\sin(225^\circ)) = -\sqrt{2} - \sqrt{2}i.$$

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$$x^2 + 2x + 5 = 0$$
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- Complex roots of quadratics always come in *conjugate pairs*: if there is one root $d + ei$, there is another $d - ei$.

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Corollary

Every polynomial with real coefficients and degree n has n complex roots (possibly including duplicates). That is, it can be factored completely into linear factors with complex number coefficients:

$$a(x - r_1)(x - r_2) \cdots (x - r_n),$$

for a real and $r_1, \ldots, r_n$ complex.

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Vieta's formulas

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- How do the coefficients of a polynomial depend on the roots?

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&= a_n(x - r_1)(x - r_2)\cdots(x - r_n) \\
&= a_n x^n - a_n(r_1 + r_2 + \cdots + r_n)x^{n-1} \\
&\quad + a_n(r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n)x^{n-2} + \cdots \\
&\quad + a_n(-1)^n(r_1 r_2 \cdots r_n)
\end{aligned}$$

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- Therefore:

$$\begin{aligned} a_{n-1}/a_n &= -(r_1 + r_2 + \cdots + r_n) \\ a_{n-2}/a_n &= r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n \\ a_{n-3}/a_n &= -(r_1 r_2 r_3 + r_1 r_2 r_4 + \cdots + r_{n-2} r_{n-1} r_n) \\ &\vdots \\ a_0/a_n &= (-1)^n r_1 r_2 \cdots r_n \end{aligned}$$

Problem

Suppose the roots of $x^3 - 2x^2 + 1$ are r_1, r_2, r_3. Find $r_1 + r_2 + r_3$ and $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$.

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- From Vieta's formulas, we know that $r_1 + r_2 + r_3$ is the negative of coefficient on x^2, so

$$r_1 + r_2 + r_3 = 2.$$

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- So
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Useful factorizations

These can often be useful:

$$x^2 - y^2 = (x - y)(x + y)$$
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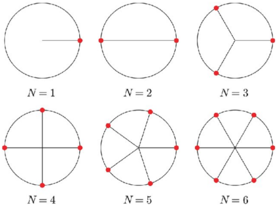

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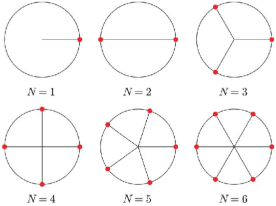

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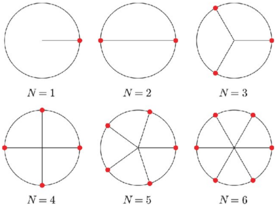

- If $n = 2$, then $\theta = 0°$ or $180°$, and so $x = \pm 1$.
- If $n = 4$, then $\theta = 0°, 90°, 180°,$ or $270°$, and so $x = 1, i, -1, -i$.

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- The roots of $x^4 + x^3 + x^2 + x + 1$ must be the same as the roots of $x^5 - 1$, leaving out $x = 1$.
- The 5th roots of unity are:

$$\cos\theta + i\sin\theta, \text{ for } \theta = 0^\circ, 72^\circ, 144^\circ, 216^\circ, 288^\circ.$$

  since $72^\circ = 360^\circ/5$.
- So answer is:

$$\cos\theta + i\sin\theta, \text{ for } \theta = 72^\circ, 144^\circ, 216^\circ, 288^\circ.$$

Next time!

Number theory