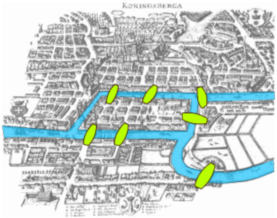
COMP 761: Lecture 9 – Graph Theory I

David Rolnick

September 23, 2020

Problem

In Königsberg (now called Kaliningrad) there are some bridges (shown below). Is it possible to cross over all of them in some order, without crossing any bridge more than once?



(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)

Course Announcements

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- Problem Set 2 will be due on Oct. 9

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- Vincent's optional list of practice problems available soon on Slack (we can give feedback on your solutions, but they will not count towards a grade)

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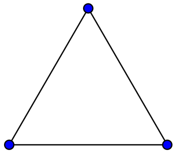
- Example of a graph: $V = \{v_1, v_2, v_3\}$ and $E = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\}$.
- Can be drawn, where the locations of the vertices don't matter:



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- And a lot more...

Special graphs: Paths and Cycles

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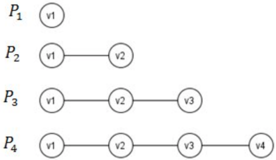
- The *path graph* P_n has n vertices $v_1, v_2, \ldots, v_n$ and $(n-1)$ edges $\{v_i, v_{i+1}\}$ for $i = 1, 2, \ldots, n-1$.



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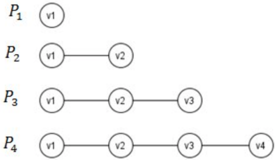
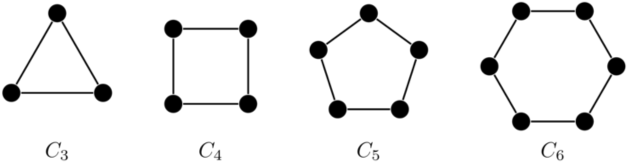


- The *cycle graph* C_n for $n \geq 3$ has n vertices $v_1, v_2, \ldots, v_n$ and n edges given by $\{v_i, v_{i+1}\} \pmod{n}$ for $i = 1, 2, \ldots, n$.



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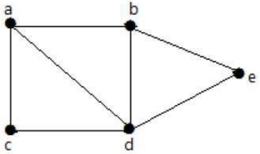

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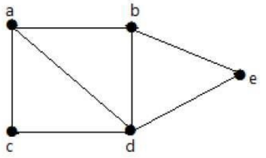

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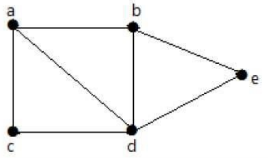

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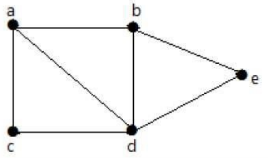

- Example:
  - The neighborhood of b is $\{a, d, e\}$.
  - The degree of b of 3 and the degree of d is 4.

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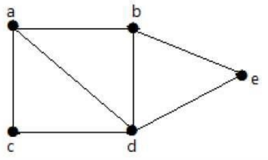
- Let's look at an example with $n = 5$ and $m = 7$.



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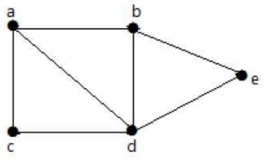

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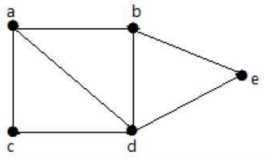

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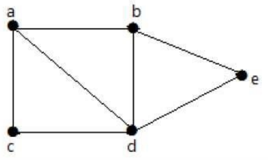

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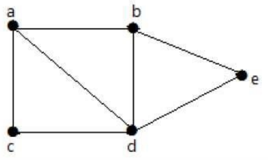

- Degrees of a, b, c, d, e are $3, 3, 2, 4, 2$.
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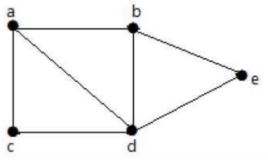

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- Why is this equal to $2m$?
- Because each edge contributes twice to the sum of degrees – once for each endpoint.
- So the sum of degrees is always twice the number of edges.

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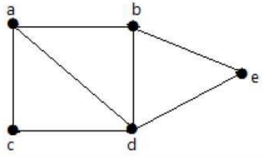

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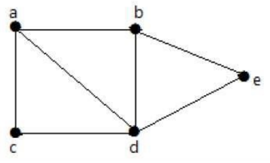
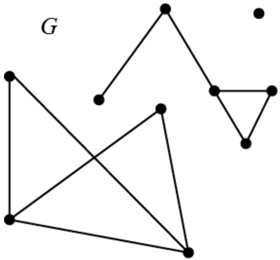


- Example (disconnected):



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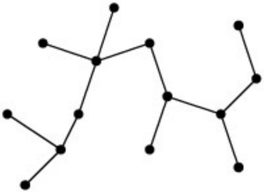

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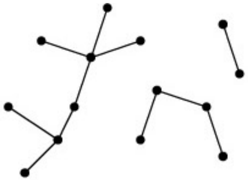

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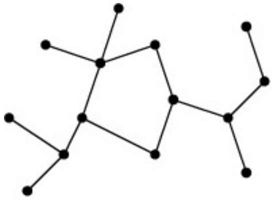

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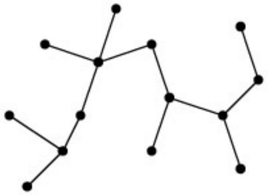

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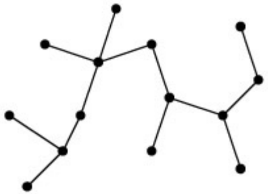
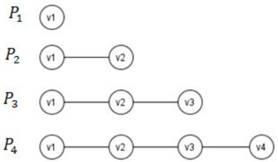


- Paths are also trees:



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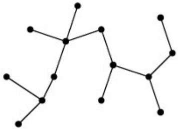

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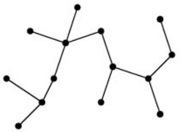

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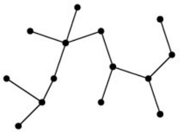

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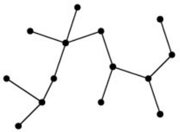

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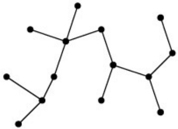

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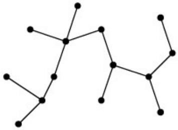

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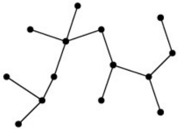

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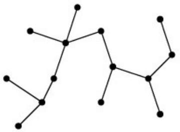

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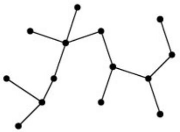

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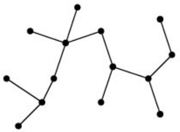

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- Or are we? What's missing?

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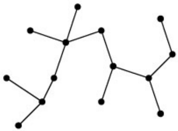

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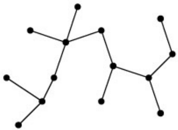

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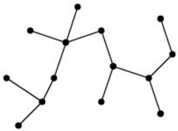

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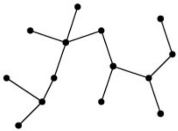

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- If $k = 1$, then the only tree has 1 vertex and 0 edges, which works.

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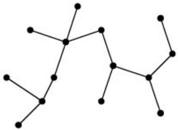

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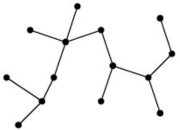

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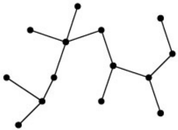

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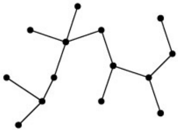

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- We'll have to stop eventually at some vertex v, with no more new vertices to go to.

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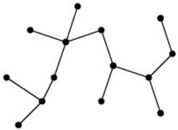

- We also assumed that T has a degree-1 vertex. Why is there one?
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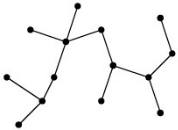

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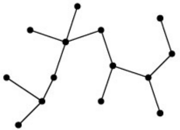

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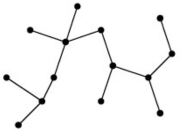

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- So v must have degree 1.
- Degree-1 vertices in a tree are called *leaves*.

Special graphs: Complete and empty graphs

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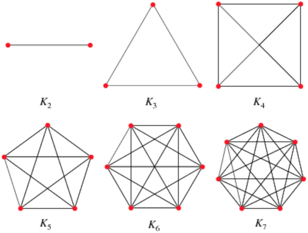

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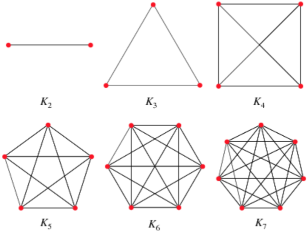

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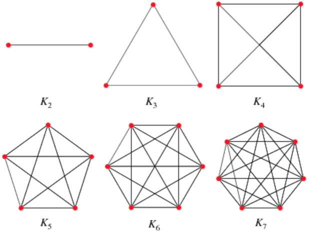

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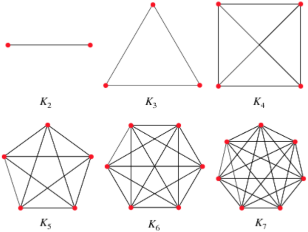

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Definitions

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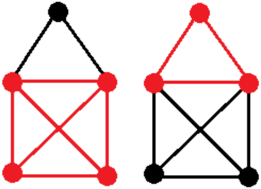

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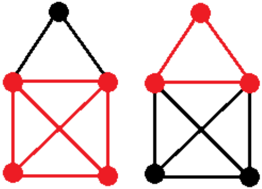

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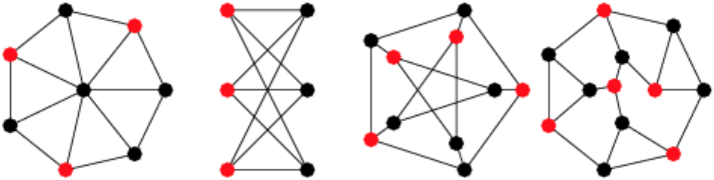

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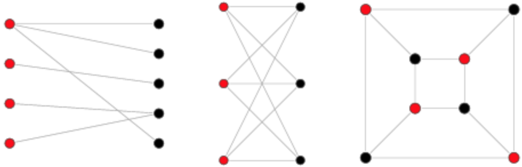

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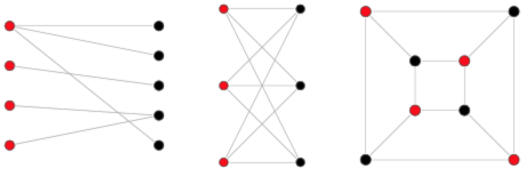

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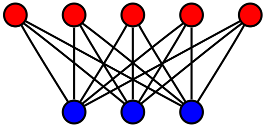

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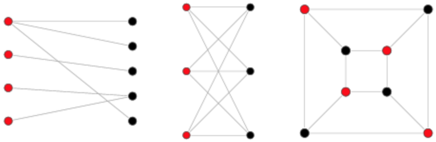

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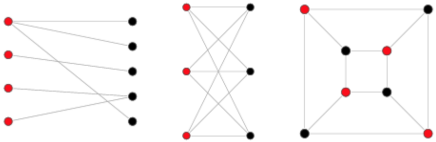

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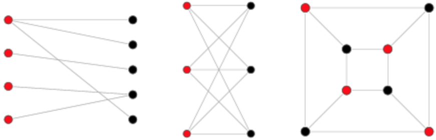

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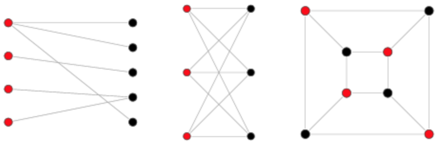

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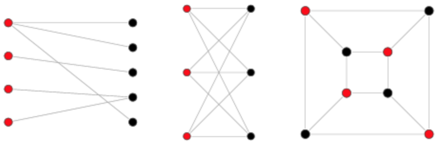

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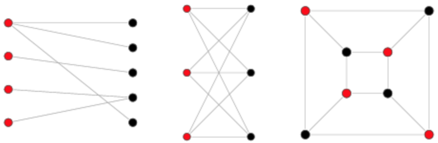

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- Therefore, the maximum number of edges is $n^2/4$.

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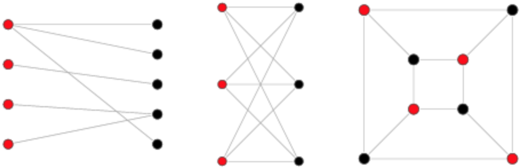
Prove that a connected graph is bipartite if and only if it doesn't have any odd-length cycles.



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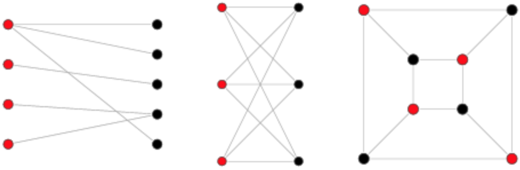

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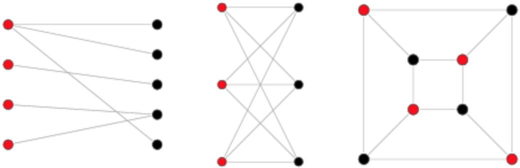

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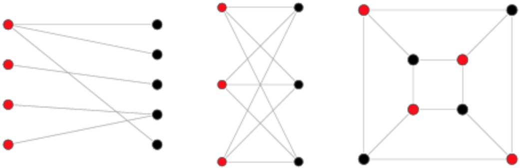

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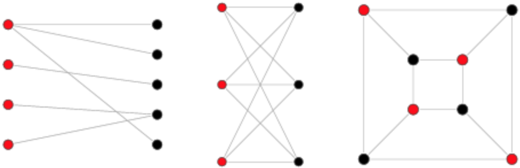

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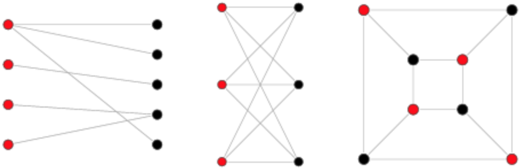

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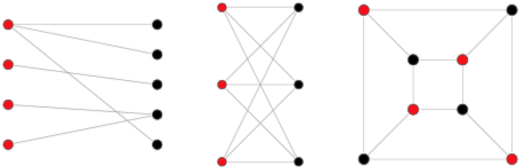

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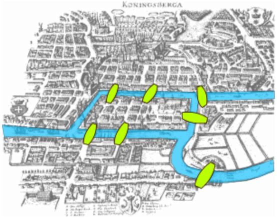
- When do we have a problem?
- If we ever have a new vertex that is connected to both vertices in parts 1 and 2.
- Can this happen?
- No. Adding a vertex with two edges to a connected graph generates a cycle, and if the edges go to parts 1 and 2 then it would have to be an odd cycle.

The Bridges of Königsberg

In Königsberg (now called Kaliningrad) there are some bridges (shown below). Is it possible to cross over all of them in some order, without crossing any bridge more than once?



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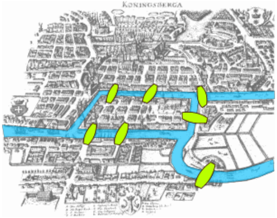

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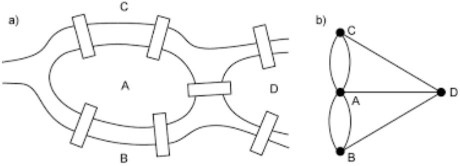

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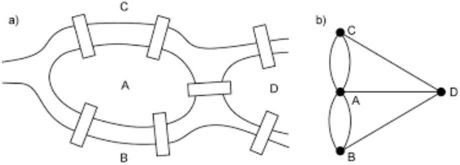

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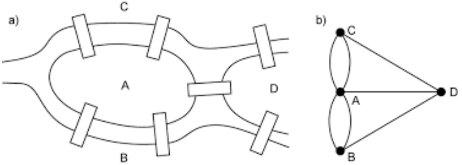
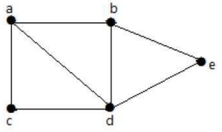


- This is actually a *multigraph*, where several edges are possible between the same pair of vertices.
- We want a path through the graph that traverses each of the edges exactly once.

Eulerian paths

In general, when does a connected graph (or a multigraph) have a path that traverses each of the edges exactly once? (Such a path is called an *Eulerian path*, named after mathematician Leonhard Euler.)



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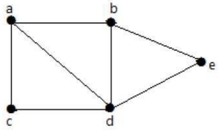

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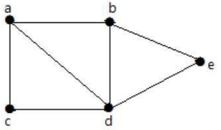

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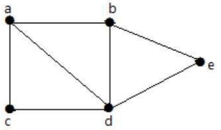

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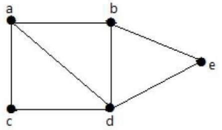

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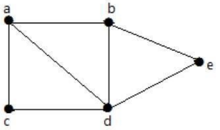

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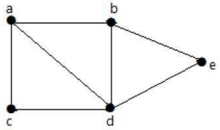
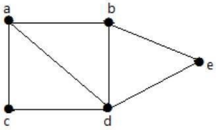


- An example of a good path: a, c, d, e, b, d, a, b.
- First, suppose that a graph has an Eulerian path. Why does it have at most 2 vertices of odd degree?

Eulerian paths

Theorem: A connected graph has an Eulerian path if and only if there are at most 2 vertices with odd degree.



- An example of a good path: a, c, d, e, b, d, a, b.
- First, suppose that a graph has an Eulerian path. Why does it have at most 2 vertices of odd degree?
- Each time a vertex occurs in the path adds 2 to its degree, unless the vertex is at the start or the end.

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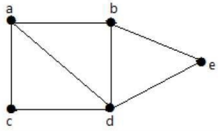

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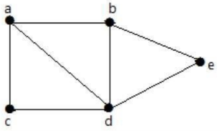

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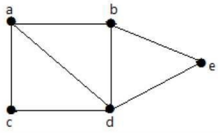
- The starting and ending vertices have odd degree – unless they are the same.

Eulerian paths



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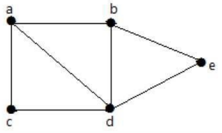

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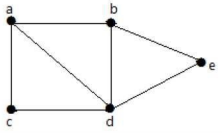
Eulerian paths



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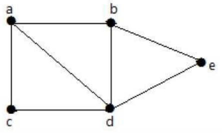
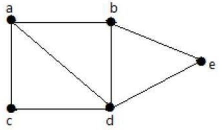


- Now suppose that the graph has at most 2 vertices of odd degree. What to do now?
- Let's divide into cases. How many vertices with odd degree could there be?
- 0 or 2, since the sum of degrees is even (twice the number of edges).

Eulerian paths



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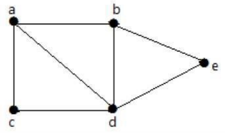
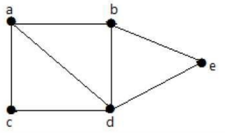


- **Case 1.** 2 vertices of odd degree: v_1 and v_2.

Eulerian paths



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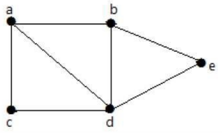

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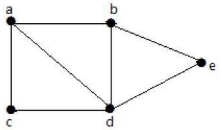

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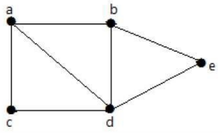

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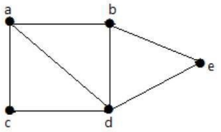

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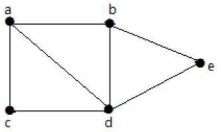

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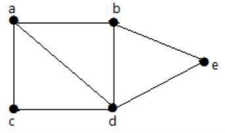
- Suppose without loss of generality that v_1 is connected to w.
- w must have even degree.
- If we take out v_1w, then we have another connected graph with 2 vertices of odd degree.
- By induction, there is an Eulerian path ending at w, and we can just add v_1w onto it.

Eulerian paths



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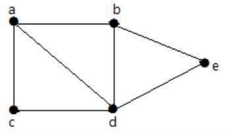

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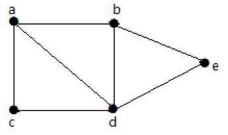
Eulerian paths



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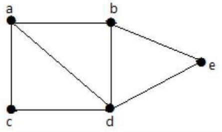

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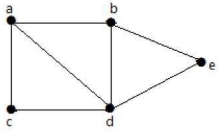

- **Case 2.** No vertices of odd degree.
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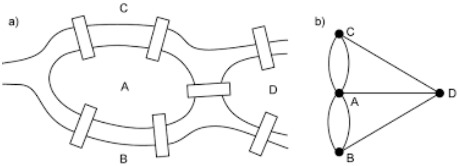
- We get a connected graph with two vertices of odd degree.
- Using Case 1, we get an Eulerian path, and can add the edge back in to complete an Eulerian path on the full graph.

The Bridges of Königsberg

In Königsberg (now called Kaliningrad) there are some bridges (shown below). Is it possible to cross over all of them in some order, without crossing any bridge more than once?



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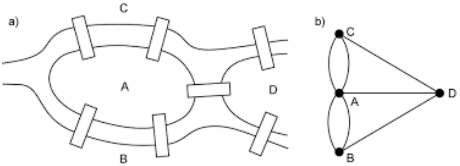

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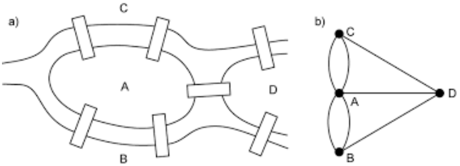

- So can we do it?
- Nope. There are more than 2 vertices with odd degree.

Other kinds of graphs

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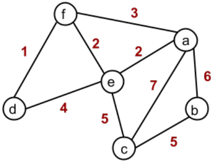
- In *weighted graphs*, each edge has a *weight* associated to it. (You can think of that like a cost associated with the edge, for example.)



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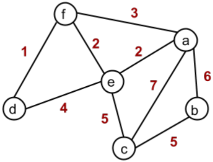

- In *directed graphs*, each edge is an ordered pair, instead of an unordered pair. You can think of each edge as having a *direction* to it.

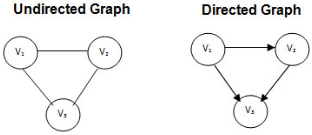

Figure 1: An Undirected Graph Figure 2: A Directed Graph

Next time!

Linear Algebra I