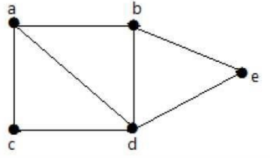
COMP 761: Lecture 12 – Graph Theory II

David Rolnick

September 30, 2020

Problem

In the following graph, I start at vertex *a* and move from where I am to an adjacent vertex each move. What is the probability of being back at *a* after 20 moves?



(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)

Course Announcements

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- Office hours this one Friday moved to 11:30 am-12:30 pm Montreal time.

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- Then, we can cancel:

$$\begin{aligned} 0 &= \lambda_k (c_1 v_1 + c_2 v_2 + \cdots + c_k v_k) - (c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \cdots + c_k \lambda_k v_k) \\ &= (\lambda_k c_1 - \lambda_1 c_1) v_1 + (\lambda_k c_2 - \lambda_2 c_2) v_2 + \cdots + (\lambda_k c_{k-1} - \lambda_{k-1} c_{k-1}) v_{k-1} \end{aligned}$$

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- Clean way to write proof - extremal principle, suppose we picked k minimal such that there is a linear dependence.

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- Therefore $\bar{\lambda} = \lambda$, so λ is actually real.

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- So $A^k v$ approaches a multiple of the *top eigenvector* v_j (the one where $|\lambda_j|$ is largest).

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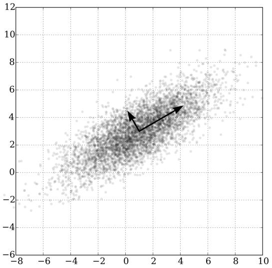

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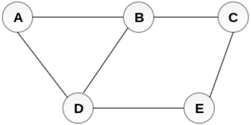

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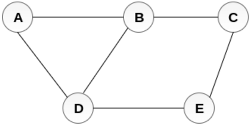

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- The *adjacency matrix* $\text{adj}(G)$ is an $n \times n$ matrix $[a_{ij}]$ where

$$a_{ij} = \begin{cases} 1 & \text{if } (i, j) \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

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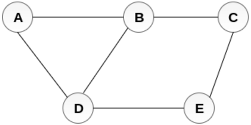

Undirected Graph

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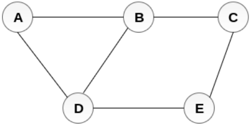

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Directed graphs

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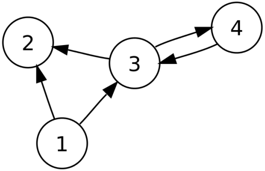
- In *directed graphs*, each edge in an *ordered* pair – e.g. $(1, 2)$ is not the same as $(2, 1)$. (Can think of edges having a *direction*.)



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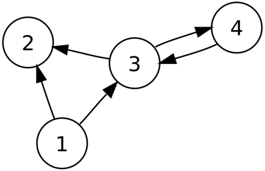

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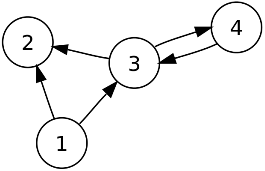

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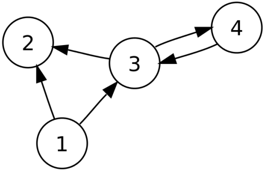

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- No longer symmetric:

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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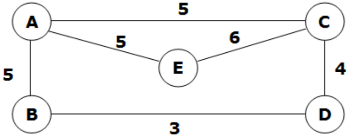

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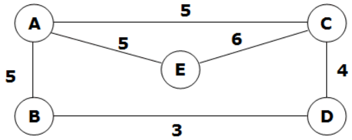

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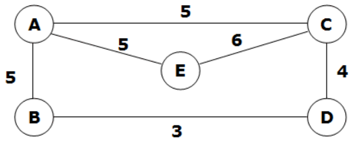

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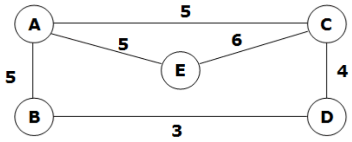

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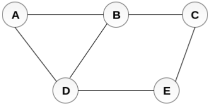
Random walks



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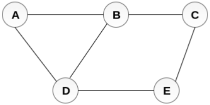

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- Often, graphs represent flow of information / objects in a network.

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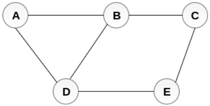

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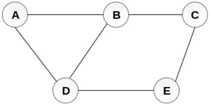

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Adjacency Matrix

- Often, graphs represent flow of information / objects in a network.
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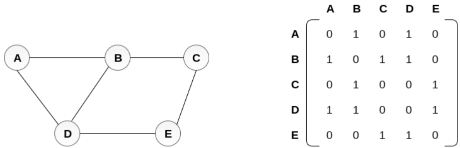
- Let's look at the square of the adjacency matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 & 2 \\ 1 & 0 & 2 & 2 & 0 \\ 1 & 1 & 2 & 3 & 0 \\ 1 & 2 & 0 & 0 & 2 \end{bmatrix}$$

Random walks



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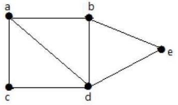
- Entry in row *i*, column *j* is $\#$ of 2-edge paths between *i* and *j*.

Random walks

In the following graph, I start at vertex *a* and move from where I am to an adjacent vertex each move. What is the probability of being back at *a* after 10 moves?



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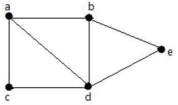

- We know that the *k*th power of the adjacency matrix *A* gives the number of *k*-edge paths between vertices.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \; A^{10} = \begin{bmatrix} 10537 & 10448 & 7801 & 12475 & 7856 \\ 10448 & 10537 & 7856 & 12475 & 7801 \\ 7801 & 7856 & 5863 & 9293 & 5829 \\ 12475 & 12475 & 9293 & 14874 & 9293 \\ 7856 & 7801 & 5829 & 9293 & 5863 \end{bmatrix}$$

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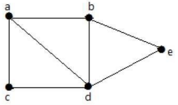

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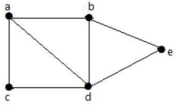

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- Answer: $10537/(10537 + 10448 + 7801 + 12475 + 7856) \approx 0.215$.

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- Therefore a long random walk on the graph will approach a limiting distribution among the vertices, given by the top eigenvector.

Diameter of a graph

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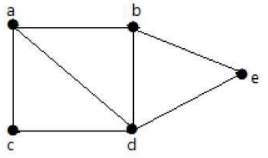
- The *distance* between two vertices in a graph is the length of the shortest path between them.



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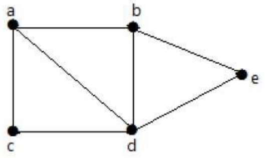

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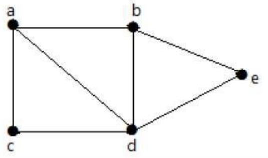

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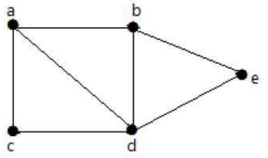

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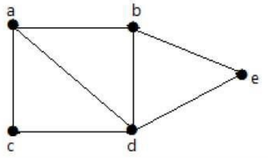

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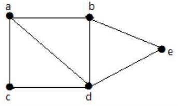
- Diameter is the lowest k such that $(\text{adj}(G))^k$ has all nonzero entries.

The Laplacian matrix



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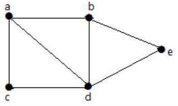

- The *Laplacian matrix* of a graph G is defined as $D - A$, where A is the adjacency matrix and D is the diagonal matrix of vertex degrees:

$$D = \begin{bmatrix} \deg(v_1) & 0 & \cdots & 0 \\ 0 & \deg(v_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \deg(v_n) \end{bmatrix}.$$

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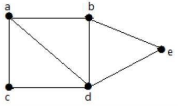

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- Example for the graph above:

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 & 0 \\ -1 & 3 & 0 & -1 & -1 \\ -1 & 0 & 2 & -1 & 0 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{bmatrix}.$$

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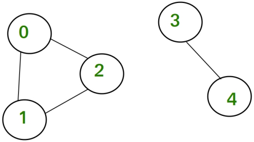
- What is one eigenvector for the Laplacian?
- The sum of each row is 0, since for every edge, there is a -1 in A and a $+1$ in D.
- That means $[1, 1, \ldots, 1]$ is an eigenvector with eigenvalue 0.

The Laplacian matrix

What if the graph is disconnected?



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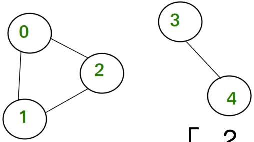

- The Laplacian for this graph is $L = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$

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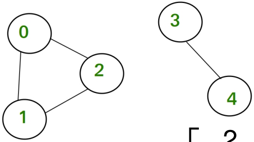

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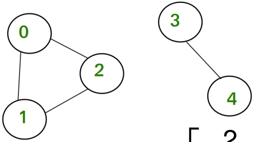

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- There are also eigenvectors $[1, 1, 1, 0, 0]$ and $[0, 0, 0, 1, 1]$, both with eigenvalue 0.

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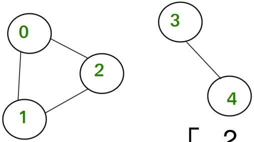

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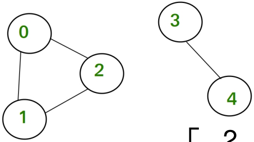
The Laplacian matrix

What if the graph is disconnected?



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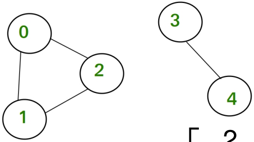

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- Won't prove it, but all eigenvalues ≥ 0 (positive semi-definite), so eigenvalue 0 is smallest.

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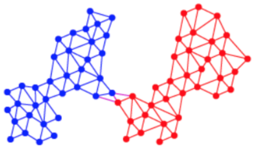

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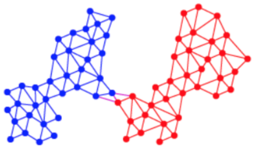

- Can formalize this by looking at min of $x^T L x$ for ± 1 vectors x.

Next time!

Calculus I