# COMP 761: Lecture 13 – Calculus I

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October 2, 2020

# Problem

Prove the AM-GM Inequality:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n}, \qquad \text{for } a_1, a_2, \ldots, a_n \geq 0$$

*(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)*

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- Find collaborators on Slack by messaging in the #problem-set-2 channel.

# The derivative

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- The derivative of a function $f(x)$ is defined as the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

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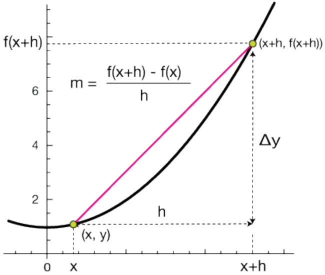
- The derivative $f'(x)$ is a different function of $x$.
- Can also write $\frac{d}{dx} f(x)$ for $f'(x)$.
- One way to think about it – instantaneous slope of a curve.



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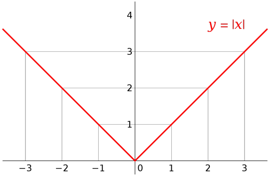


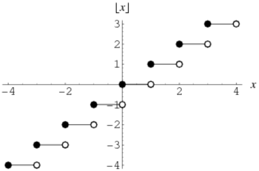

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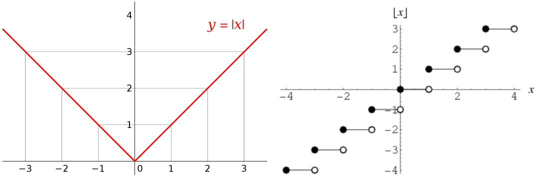


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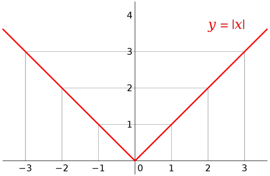


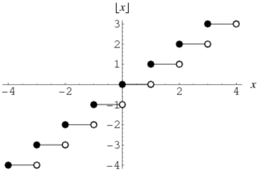


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- Products, sums, powers, compositions of differentiable functions are differentiable.

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- So the answer is $\binom{n}{1}x^{n-1} = nx^{n-1}$.

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- (Terminology: "differentiate" = "take the derivative of")
- Example:

$$\begin{aligned}\frac{d}{dx}(5x^4 - 2x^3 + 1) &= 5\frac{d}{dx}x^4 - 2\frac{d}{dx}x^3 + \frac{d}{dx}1 \\ &= 5(4x^3) - 2(3x^2) + 0 \\ &= 20x^3 - 6x^2.\end{aligned}$$

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- Example:

$$\frac{d}{dx}(x^m)(x^n) = (mx^{m-1})(x^n)+(x^m)(nx^{n-1}) = (m+n)x^{m+n-1}.$$

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- Makes sense since this is just $\frac{d}{dx}x^m = mx^{m-1}$ for $m = -n$.
- In fact, $\frac{d}{dx}x^m = mx^{m-1}$ is true for all real $m$.

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$$\begin{aligned}\frac{d}{dx}(x^2+1)^3 &= \left(\frac{d}{dx}(x^2+1)\right)\left(3(x^2+1)^2\right)\\ &= (2x)\cdot 3(x^2+1)^2\\ &= 6x(x^2+1)^2.\end{aligned}$$

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- So what is is $\lim_{h\to 0}\frac{c^h-1}{h}$?
- We can define the number $e \approx 2.718\ldots$ to be the value of $c$ for which this limit equals 1.
- Then, $e^x$ has derivative just $e^x$.
- $e$ turns up everywhere, just like $\pi$.

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- We know $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$.
- So setting $n = 1/h$, we have

$$\lim_{n \to \infty} \frac{e^{1/n} - 1}{1/n} = 1.$$

# More about $e$

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- From this, we get another definition of $e$:

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

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- So $f(e^t)\cdot e^t = 1$.
- That means that $f(e^t) = 1/e^t$.
- Therefore, for any $x > 0$, we have

$$\frac{d}{dx}\log(x) = f(x) = \frac{1}{x}.$$

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- Measure the rate of change of the rate of change.
- For example, acceleration is the derivative of velocity, which is the derivative of position.

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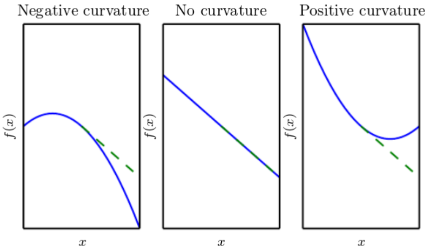

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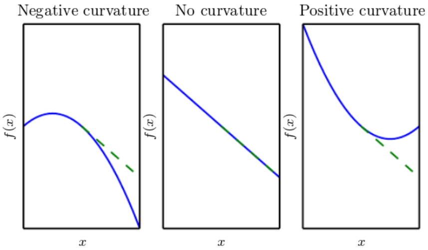


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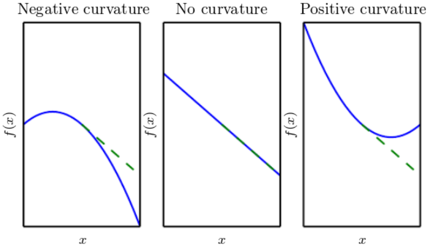


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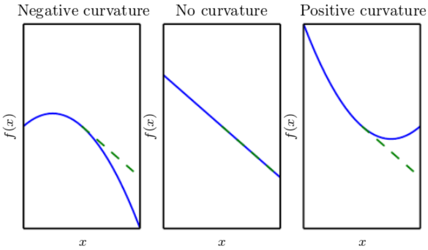


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- If a function $f(x)$ has $f''(x) \leq 0$ for all $x$, we say it is *concave*.
- **Most functions are neither convex nor concave.**

# Next time!

**Calculus II**