COMP 761: Lecture 15 – Calculus III

David Rolnick

October 7, 2020

Problem

Prove: $1/e = {}^1\!/_{2!} - {}^1\!/_{3!} + {}^1\!/_{4!} - {}^1\!/_{5!} + \cdots$.

(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)

Course Announcements

Course Announcements

- Simplifying Problem 5: You can assume the eigenvalues of the circulant matrix are all different (this is the case for almost all circulant matrices). Now the thing about the Vandermonde matrix is not necessary.

Course Announcements

- Simplifying Problem 5: You can assume the eigenvalues of the circulant matrix are all different (this is the case for almost all circulant matrices). Now the thing about the Vandermonde matrix is not necessary.
- Office hours: Vincent on Thu at 10:30 am, David on Fri at 10 am

Derivative of a vector by a vector

Derivative of a vector by a vector

- The partial derivatives of a vector y by a vector x can be represented in a matrix called the *Jacobian*:

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \partial y_1 / \partial x_1 & \partial y_1 / \partial x_2 & \cdots & \partial y_1 / \partial x_n \\ \partial y_2 / \partial x_1 & \partial y_2 / \partial x_2 & \cdots & \partial y_2 / \partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial y_m / \partial x_1 & \partial y_m / \partial x_2 & \cdots & \partial y_m / \partial x_n \end{bmatrix}$$

Derivative of a vector by a vector

- The partial derivatives of a vector y by a vector x can be represented in a matrix called the *Jacobian*:

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 & \cdots & \partial y_1/\partial x_n \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 & \cdots & \partial y_2/\partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial y_m/\partial x_1 & \partial y_m/\partial x_2 & \cdots & \partial y_m/\partial x_n \end{bmatrix}$$

- What is the partial derivative $\frac{\partial}{\partial x} Ax$?

Derivative of a vector by a vector

- The partial derivatives of a vector y by a vector x can be represented in a matrix called the *Jacobian*:

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 & \cdots & \partial y_1/\partial x_n \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 & \cdots & \partial y_2/\partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial y_m/\partial x_1 & \partial y_m/\partial x_2 & \cdots & \partial y_m/\partial x_n \end{bmatrix}$$

- What is the partial derivative $\frac{\partial}{\partial x} Ax$?
- If $y = Ax$, then we have:

$$y_i = \sum_{j=1}^{n} a_{ij} x_j.$$

Derivative of a vector by a vector

- The partial derivatives of a vector y by a vector x can be represented in a matrix called the *Jacobian*:

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 & \cdots & \partial y_1/\partial x_n \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 & \cdots & \partial y_2/\partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial y_m/\partial x_1 & \partial y_m/\partial x_2 & \cdots & \partial y_m/\partial x_n \end{bmatrix}$$

- What is the partial derivative $\frac{\partial}{\partial x} Ax$?
- If $y = Ax$, then we have:

$$y_i = \sum_{j=1}^{n} a_{ij} x_j.$$

- So $\partial y_i/\partial x_j = a_{ij}$.

Derivative of a vector by a vector

- The partial derivatives of a vector y by a vector x can be represented in a matrix called the *Jacobian*:

$$\frac{\partial y}{\partial x} = \begin{bmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 & \cdots & \partial y_1/\partial x_n \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 & \cdots & \partial y_2/\partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial y_m/\partial x_1 & \partial y_m/\partial x_2 & \cdots & \partial y_m/\partial x_n \end{bmatrix}$$

- What is the partial derivative $\frac{\partial}{\partial x} Ax$?
- If $y = Ax$, then we have:

$$y_i = \sum_{j=1}^{n} a_{ij} x_j.$$

- So $\partial y_i/\partial x_j = a_{ij}$.
- Therefore:

$$\frac{\partial}{\partial x} Ax = A.$$

Derivative of a vector by a vector

Derivative of a vector by a vector

- If the vector y is a function of the vector x, which is itself a function of t, then we can use the chain rule for each y_k:

$$\frac{dy_k}{dt} = \frac{\partial y_k}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial y_k}{\partial x_n}\frac{dx_n}{dt}.$$

Derivative of a vector by a vector

- If the vector y is a function of the vector x, which is itself a function of t, then we can use the chain rule for each y_k:

$$\frac{dy_k}{dt} = \frac{\partial y_k}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial y_k}{\partial x_n}\frac{dx_n}{dt}.$$

- Therefore:

$$\frac{dy}{dt} = \begin{bmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 & \cdots & \partial y_1/\partial x_n \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 & \cdots & \partial y_2/\partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial y_m/\partial x_1 & \partial y_m/\partial x_2 & \cdots & \partial y_m/\partial x_n \end{bmatrix} \begin{bmatrix} dx_1/dt \\ dx_2/dt \\ \vdots \\ dx_n/dt \end{bmatrix}.$$

Derivative of a scalar by a matrix

Derivative of a scalar by a matrix

- If we take the derivative of scalar y by matrix X, we get a matrix:

$$\frac{\partial y}{\partial X} = \begin{bmatrix} \partial y/\partial x_{11} & \partial y/\partial x_{12} & \cdots & \partial y/\partial x_{1n} \\ \partial y/\partial x_{21} & \partial y/\partial x_{22} & \cdots & \partial y/\partial x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \partial y/\partial x_{m1} & \partial y/\partial x_{m2} & \cdots & \partial y/\partial x_{mn} \end{bmatrix}$$

Derivative of a scalar by a matrix

- If we take the derivative of scalar y by matrix X, we get a matrix:

$$\frac{\partial y}{\partial X} = \begin{bmatrix} \partial y/\partial x_{11} & \partial y/\partial x_{12} & \cdots & \partial y/\partial x_{1n} \\ \partial y/\partial x_{21} & \partial y/\partial x_{22} & \cdots & \partial y/\partial x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \partial y/\partial x_{m1} & \partial y/\partial x_{m2} & \cdots & \partial y/\partial x_{mn} \end{bmatrix}$$

- What is the derivative $\frac{\partial}{\partial X}\operatorname{tr}(X)$ for square matrix X?

Derivative of a scalar by a matrix

- If we take the derivative of scalar y by matrix X, we get a matrix:

$$\frac{\partial y}{\partial X} = \begin{bmatrix} \partial y / \partial x_{11} & \partial y / \partial x_{12} & \cdots & \partial y / \partial x_{1n} \\ \partial y / \partial x_{21} & \partial y / \partial x_{22} & \cdots & \partial y / \partial x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \partial y / \partial x_{m1} & \partial y / \partial x_{m2} & \cdots & \partial y / \partial x_{mn} \end{bmatrix}$$

- What is the derivative $\frac{\partial}{\partial X} \operatorname{tr}(X)$ for square matrix X?
- $\operatorname{tr}(X) = \sum_{i=1}^{n} x_{ii}$.

Derivative of a scalar by a matrix

- If we take the derivative of scalar y by matrix X, we get a matrix:

$$\frac{\partial y}{\partial X} = \begin{bmatrix} \partial y / \partial x_{11} & \partial y / \partial x_{12} & \cdots & \partial y / \partial x_{1n} \\ \partial y / \partial x_{21} & \partial y / \partial x_{22} & \cdots & \partial y / \partial x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \partial y / \partial x_{m1} & \partial y / \partial x_{m2} & \cdots & \partial y / \partial x_{mn} \end{bmatrix}$$

- What is the derivative $\frac{\partial}{\partial X}\operatorname{tr}(X)$ for square matrix X?
- $\operatorname{tr}(X) = \sum_{i=1}^{n} x_{ii}$.
- So derivative with respect to x_{ij} is 0 unless $i = j$.

Derivative of a scalar by a matrix

- If we take the derivative of scalar y by matrix X, we get a matrix:

$$\frac{\partial y}{\partial X} = \begin{bmatrix} \partial y/\partial x_{11} & \partial y/\partial x_{12} & \cdots & \partial y/\partial x_{1n} \\ \partial y/\partial x_{21} & \partial y/\partial x_{22} & \cdots & \partial y/\partial x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \partial y/\partial x_{m1} & \partial y/\partial x_{m2} & \cdots & \partial y/\partial x_{mn} \end{bmatrix}$$

- What is the derivative $\frac{\partial}{\partial X}\operatorname{tr}(X)$ for square matrix X?
- $\operatorname{tr}(X) = \sum_{i=1}^{n} x_{ii}$.
- So derivative with respect to x_{ij} is 0 unless $i = j$.
- Derivative with respect to x_{ii} is 1.

Derivative of a scalar by a matrix

- If we take the derivative of scalar y by matrix X, we get a matrix:

$$\frac{\partial y}{\partial X} = \begin{bmatrix} \partial y/\partial x_{11} & \partial y/\partial x_{12} & \cdots & \partial y/\partial x_{1n} \\ \partial y/\partial x_{21} & \partial y/\partial x_{22} & \cdots & \partial y/\partial x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \partial y/\partial x_{m1} & \partial y/\partial x_{m2} & \cdots & \partial y/\partial x_{mn} \end{bmatrix}$$

- What is the derivative $\frac{\partial}{\partial X}\operatorname{tr}(X)$ for square matrix X?
- $\operatorname{tr}(X) = \sum_{i=1}^{n} x_{ii}$.
- So derivative with respect to x_{ij} is 0 unless $i = j$.
- Derivative with respect to x_{ii} is 1.
- Therefore:

$$\frac{\partial}{\partial X}\operatorname{tr}(X) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I.$$

Derivative of a scalar by a matrix

Derivative of a scalar by a matrix

- What is the derivative $\frac{\partial}{\partial X} \operatorname{tr}(XA)$ for matrices $X \in \mathbb{R}^{m \times n}$, $A \in \mathbb{R}^{n \times m}$?

Derivative of a scalar by a matrix

- What is the derivative $\frac{\partial}{\partial X}\operatorname{tr}(XA)$ for matrices $X \in \mathbb{R}^{m\times n}$, $A \in \mathbb{R}^{n\times m}$?
- For $1 \leq i \leq m$, the ith diagonal entry of XA is:

$$\sum_{j=1}^{n} x_{ij}a_{ji},$$

Derivative of a scalar by a matrix

- What is the derivative $\frac{\partial}{\partial X}\operatorname{tr}(XA)$ for matrices $X \in \mathbb{R}^{m\times n}$, $A \in \mathbb{R}^{n\times m}$?
- For $1 \le i \le m$, the ith diagonal entry of XA is:

$$\sum_{j=1}^{n} x_{ij} a_{ji},$$

- So we have:

$$\operatorname{tr}(XA) = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} a_{ji}.$$

Derivative of a scalar by a matrix

- What is the derivative $\frac{\partial}{\partial X}\operatorname{tr}(XA)$ for matrices $X \in \mathbb{R}^{m\times n}$, $A \in \mathbb{R}^{n\times m}$?
- For $1 \leq i \leq m$, the ith diagonal entry of XA is:

$$\sum_{j=1}^{n} x_{ij} a_{ji},$$

- So we have:

$$\operatorname{tr}(XA) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} a_{ji}.$$

- Therefore, the partial derivative of $\operatorname{tr}(XA)$ by x_{ij} is a_{ji}, giving us:

$$\frac{\partial}{\partial X}\operatorname{tr}(XA) = A^T.$$

Problem

Let X, A be $n \times n$ matrices, with X the matrix with all entries equal to t^2. What is the derivative $\frac{d}{dt}\,\mathrm{tr}(XA)$?

Problem

Let X, A be $n \times n$ matrices, with X the matrix with all entries equal to t^2. What is the derivative $\frac{d}{dt}\operatorname{tr}(XA)$?

- We know that:

$$\frac{\partial}{\partial X}\operatorname{tr}(XA) = A^T.$$

Problem

Let X, A be $n \times n$ matrices, with X the matrix with all entries equal to t^2. What is the derivative $\frac{d}{dt}\operatorname{tr}(XA)$?

- We know that:

$$\frac{\partial}{\partial X}\operatorname{tr}(XA) = A^T.$$

- By the chain rule, we have:

$$\begin{aligned}\frac{d}{dt}\operatorname{tr}(XA) &= \sum_{i,j}\left(\frac{\partial}{\partial x_{ij}}\operatorname{tr}(XA)\right)\left(\frac{dx_{ij}}{dt}\right)\\ &= \sum_{i,j}(a_{ji})\,(2t)\,.\end{aligned}$$

Problem

Let X, A be $n \times n$ matrices, with X the matrix with all entries equal to t^2. What is the derivative $\frac{d}{dt}\operatorname{tr}(XA)$?

- We know that:
$$\frac{\partial}{\partial X}\operatorname{tr}(XA) = A^T.$$
- By the chain rule, we have:
$$\frac{d}{dt}\operatorname{tr}(XA) = \sum_{i,j}\left(\frac{\partial}{\partial x_{ij}}\operatorname{tr}(XA)\right)\left(\frac{dx_{ij}}{dt}\right)$$
$$= \sum_{i,j}(a_{ji})\,(2t)\,.$$
- So the answer is just $2t$ times the sum of entries in A.

Infinite series

Infinite series

- Sometimes, an infinite series has a well-defined sum.

Infinite series

- Sometimes, an infinite series has a well-defined sum.
- Example, what is:

$$\sum_{n=1}^{\infty} 1/2^n = 1 + 1/2 + 1/4 + 1/8 + \cdots ?$$

Infinite series

- Sometimes, an infinite series has a well-defined sum.
- Example, what is:

$$\sum_{n=1}^{\infty} 1/2^n = 1 + 1/2 + 1/4 + 1/8 + \cdots ?$$

- We have

$$\begin{aligned} 1 - 1/2 &= 1/2 \\ 1 - (1/2 + 1/4) &= 1/4 \\ 1 - (1/2 + 1/4 + 1/8) &= 1/8 \\ &\vdots \end{aligned}$$

Infinite series

- Sometimes, an infinite series has a well-defined sum.
- Example, what is:

$$\sum_{n=1}^{\infty} 1/2^n = 1 + 1/2 + 1/4 + 1/8 + \cdots ?$$

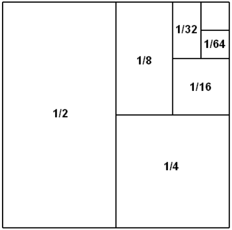
- We have

$$\begin{aligned} 1 - 1/2 &= 1/2 \\ 1 - (1/2 + 1/4) &= 1/4 \\ 1 - (1/2 + 1/4 + 1/8) &= 1/8 \\ &\vdots \end{aligned}$$

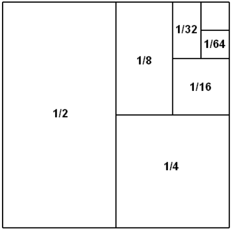
- So makes sense that $1/2 + 1/4 + 1/8 + \cdots = 1$ and therefore

$$1 + 1/2 + 1/4 + 1/8 + \cdots = 2.$$

Infinite series



Infinite series



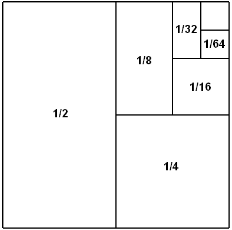
- Another argument

$$1 + x + \cdots + x^{n-1} + x^n = \frac{1 - x^{n+1}}{1 - x},$$

so

$$\lim_{n\to\infty} (1 + x + \cdots + x^{n-1} + x^n) = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1 - x}, \text{ if } |x| < 1.$$

Infinite series



- Another argument

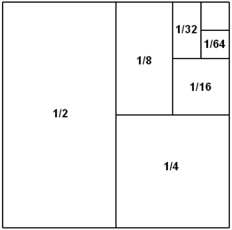
$$1 + x + \cdots + x^{n-1} + x^n = \frac{1 - x^{n+1}}{1 - x},$$

so

$$\lim_{n\to\infty} (1 + x + \cdots + x^{n-1} + x^n) = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1 - x}, \text{ if } |x| < 1.$$

- *This only works if $|x| < 1$.*

Infinite series



- Another argument

$$1 + x + \cdots + x^{n-1} + x^n = \frac{1 - x^{n+1}}{1 - x},$$

so

$$\lim_{n\to\infty}(1 + x + \cdots + x^{n-1} + x^n) = \lim_{n\to\infty}\frac{1 - x^{n+1}}{1 - x} = \frac{1}{1 - x}, \text{ if } |x| < 1.$$

- *This only works if* $|x| < 1$.
- If $x = 1/2$, then $1 + {}^1/_2 + {}^1/_4 + {}^1/_8 + \cdots = \frac{1}{1 - {}^1/_2} = 2$.

Infinite series

Infinite series

- What about if $x = -1$?

$$1 + x + x^2 + \cdots = 1 - 1 + 1 - 1 + 1 - \cdots .$$

Infinite series

- What about if $x = -1$?

$$1 + x + x^2 + \cdots = 1 - 1 + 1 - 1 + 1 - \cdots .$$

- Partial sums:

$$\begin{aligned} 1 &= 1 \\ 1 - 1 &= 0 \\ 1 - 1 + 1 &= 1 \\ 1 - 1 + 1 - 1 &= 0 \\ &\vdots \end{aligned}$$

Infinite series

- What about if $x = -1$?

$$1 + x + x^2 + \cdots = 1 - 1 + 1 - 1 + 1 - \cdots .$$

- Partial sums:

$$\begin{aligned} 1 &= 1 \\ 1 - 1 &= 0 \\ 1 - 1 + 1 &= 1 \\ 1 - 1 + 1 - 1 &= 0 \\ &\vdots \end{aligned}$$

- Doesn't converge to a single sum, so the limit $\lim_{n\to\infty} \sum_{k=0}^{n} x^k$ doesn't actually exist for $x = -1$.

Taylor series

Taylor series

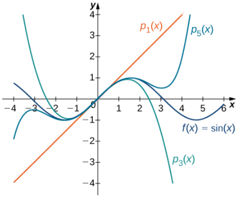
- We know that $f(x) \approx f(0) + f'(0)x$ for x close to 0.

Taylor series

- We know that $f(x) \approx f(0) + f'(0)x$ for x close to 0.
- Can think of this as a way to approximate $f(x)$ around $x = 0$ by a polynomial of degree 1.

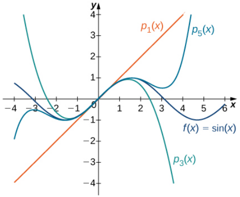
Taylor series

- We know that $f(x) \approx f(0) + f'(0)x$ for x close to 0.
- Can think of this as a way to approximate $f(x)$ around $x = 0$ by a polynomial of degree 1.
- What about approximation by polynomials of higher degree?



Taylor series

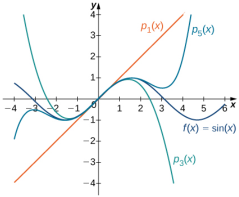
- We know that $f(x) \approx f(0) + f'(0)x$ for x close to 0.
- Can think of this as a way to approximate $f(x)$ around $x = 0$ by a polynomial of degree 1.
- What about approximation by polynomials of higher degree?



- What degree 2 polynomial $p_2(x)$ approximates $f(x)$ near $x = 0$?

Taylor series

- We know that $f(x) \approx f(0) + f'(0)x$ for x close to 0.
- Can think of this as a way to approximate $f(x)$ around $x = 0$ by a polynomial of degree 1.
- What about approximation by polynomials of higher degree?



- What degree 2 polynomial $p_2(x)$ approximates $f(x)$ near $x = 0$?
- Want to have

$$p_2(0) = f(0)$$
$$p_2'(0) = f'(0)$$
$$p_2''(0) = f''(0)$$

Taylor series

- Want to have

$$p_2(0) = f(0)$$
$$p_2'(0) = f'(0)$$
$$p_2''(0) = f''(0)$$

Taylor series

- Want to have

$$p_2(0) = f(0)$$
$$p_2'(0) = f'(0)$$
$$p_2''(0) = f''(0)$$

- Let $p_2(x) = c + bx + ax^2$.

Taylor series

- Want to have

$$p_2(0) = f(0)$$
$$p_2'(0) = f'(0)$$
$$p_2''(0) = f''(0)$$

- Let $p_2(x) = c + bx + ax^2$.
- We have $p_2'(x) = b + 2ax$, so $p_2'(0) = b$.

Taylor series

- Want to have

$$\begin{aligned} p_2(0) &= f(0) \\ p_2'(0) &= f'(0) \\ p_2''(0) &= f''(0) \end{aligned}$$

- Let $p_2(x) = c + bx + ax^2$.
- We have $p_2'(x) = b + 2ax$, so $p_2'(0) = b$.
- We have $p_2''(x) = 2a$, so $p_2''(0) = 2a$.

Taylor series

- Want to have

$$p_2(0) = f(0)$$
$$p_2'(0) = f'(0)$$
$$p_2''(0) = f''(0)$$

- Let $p_2(x) = c + bx + ax^2$.
- We have $p_2'(x) = b + 2ax$, so $p_2'(0) = b$.
- We have $p_2''(x) = 2a$, so $p_2''(0) = 2a$.
- We therefore want $b = f'(0)$ and $a = f''(0)/2$.

$$p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2.$$

Taylor series

Taylor series

- We can do the same thing to find a degree-3 polynomial $p_3(x)$ approximating $f(x)$ up to third deriviatives:

$$p_3(0) = f(0)$$
$$p_3'(0) = f'(0)$$
$$p_3''(0) = f''(0)$$
$$p_3'''(0) = f'''(0)$$

- Let $p_3(x) = d + cx + bx^2 + ax^3$.

Taylor series

- We can do the same thing to find a degree-3 polynomial $p_3(x)$ approximating $f(x)$ up to third deriviatives:

$$p_3(0) = f(0)$$
$$p_3'(0) = f'(0)$$
$$p_3''(0) = f''(0)$$
$$p_3'''(0) = f'''(0)$$

- Let $p_3(x) = d + cx + bx^2 + ax^3$.
- We have $p_3'(x) = c + 2bx + 3ax^2$, so $p_3'(0) = c$.

Taylor series

- We can do the same thing to find a degree-3 polynomial $p_3(x)$ approximating $f(x)$ up to third deriviatives:

$$\begin{aligned} p_3(0) &= f(0) \\ p_3'(0) &= f'(0) \\ p_3''(0) &= f''(0) \\ p_3'''(0) &= f'''(0) \end{aligned}$$

- Let $p_3(x) = d + cx + bx^2 + ax^3$.
- We have $p_3'(x) = c + 2bx + 3ax^2$, so $p_3'(0) = c$.
- We have $p_3''(x) = 2b + 6ax$, so $p_3''(0) = 2b$.
- We have $p_3'''(x) = 6a$, so $p_3'''(0) = 6a$.

Taylor series

- We can do the same thing to find a degree-3 polynomial $p_3(x)$ approximating $f(x)$ up to third deriviatives:

$$p_3(0) = f(0)$$
$$p_3'(0) = f'(0)$$
$$p_3''(0) = f''(0)$$
$$p_3'''(0) = f'''(0)$$

- Let $p_3(x) = d + cx + bx^2 + ax^3$.
- We have $p_3'(x) = c + 2bx + 3ax^2$, so $p_3'(0) = c$.
- We have $p_3''(x) = 2b + 6ax$, so $p_3''(0) = 2b$.
- We have $p_3'''(x) = 6a$, so $p_3'''(0) = 6a$.
- We therefore want $c = f'(0)$ and $b = f''(0)/2$ and $a = f'''(0)/6$.

$$p_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3.$$

Taylor series

Taylor series

- More generally, we can find a degree-k polynomial $p_k(x)$ approximating $f(x)$ up to kth derivatives.

$$p_1(x) = f(0) + f'(0)x$$

$$p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

$$p_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3$$

$$\vdots$$

$$p_k(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(k)}(0)}{k!}x^k.$$

Taylor series

- More generally, we can find a degree-k polynomial $p_k(x)$ approximating $f(x)$ up to kth deriviatives.

$$p_1(x) = f(0) + f'(0)x$$

$$p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

$$p_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3$$

$$\vdots$$

$$p_k(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(k)}(0)}{k!}x^k.$$

- The limit of infinite degree is the *Taylor series*:

$$p_\infty(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \cdots.$$

Taylor series

- More generally, we can find a degree-k polynomial $p_k(x)$ approximating $f(x)$ up to kth deriviatives.

$$p_1(x) = f(0) + f'(0)x$$
$$p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$
$$p_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3$$
$$\vdots$$
$$p_k(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(k)}(0)}{k!}x^k.$$

- The limit of infinite degree is the *Taylor series*:

$$p_\infty(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \cdots .$$

- At least for x close to 0, we have $f(x) = p_\infty(x)$.

Taylor series

Taylor series

- The Taylor series for f about $x = 0$:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \cdots .$$

Taylor series

- The Taylor series for f about $x = 0$:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \cdots.$$

- Can similarly do this around another point $x = a$:

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots.$$

Taylor series

- The Taylor series for f about $x = 0$:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \cdots .$$

- Can similarly do this around another point $x = a$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots .$$

- As long as the infinite sum is convergent, i.e. the limit is a well-defined number, that number equals $f(x)$.
- For some functions $f(x)$, it converges for all $x \in \mathbb{R}$.
- For other functions, converges just for x in an interval e.g. $|x| < R$.

Next time!

Probability I