COMP 761: Lecture 16 – Taylor Series

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Problem

What is $\lim_{x \to 0} \frac{e^x - 1}{\sin(x)}$?

(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)

Course Announcements

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- Problem Set 2 is due today at 11:59 pm Montreal

Course Announcements

- Problem Set 2 is due today at 11:59 pm Montreal
- No class on Monday for Thanksgiving!

Taylor series

Taylor series

- We can find a degree-k polynomial $p_k(x)$ approximating $f(x)$ up to kth deriviatives.

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\begin{aligned}
p_1(x) &= f(0) + f'(0)x \\
p_2(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 \\
p_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 \\
&\vdots \\
p_k(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(k)}(0)}{k!}x^k.
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$$p_k(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(k)}(0)}{k!}x^k.$$

- The limit of infinite degree is the *Taylor series*:

$$p_\infty(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \cdots.$$

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- At least for x close to 0, we have $f(x) = p_\infty(x)$.

Taylor series

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- The Taylor series for f about $x = 0$:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \cdots .$$

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- Can similarly do this around another point $x = a$:

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots .$$

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- As long as the infinite sum is convergent, i.e. the limit is a well-defined number, that number equals $f(x)$.
- For some functions $f(x)$, it converges for all $x \in \mathbb{R}$.
- For other functions, converges just for x in an interval e.g. $|x| < R$.

Taylor series: e^x

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- What is the Taylor series for e^x (around $x = 0$)?
- For $f(x) = e^x$, we have $f(x) = f'(x) = f''(x) = \cdots = e^x$, so

$$f(0) = f'(0) = f''(0) = \cdots = e^0 = 1.$$

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- Therefore the Taylor series is:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots .$$

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- Special cases:

$$e = 1 + 1 + {}^1/_{2!} + {}^1/_{3!} + {}^1/_{4!} + {}^1/_{5!} + \cdots .$$
$$e^{-1} = 1 - 1 + {}^1/_{2!} - {}^1/_{3!} + {}^1/_{4!} - {}^1/_{5!} + \cdots .$$

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- What is the Taylor series (at $x = 0$) for a polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x_1 + a_0?$$

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- We have:

$$p(0) = a_0$$
$$p'(0) = (na_n x^{n-1} + \cdots + 3a_3 x^2 + 2a_2 x + a_1)|_{x=0} = a_1$$
$$p''(0) = (n(n-1)a_n x^{n-2} + \cdots + (3 \cdot 2)a_3 x + 2a_2)|_{x=0} = 2a_2$$
$$p'''(0) = (n(n-1)(n-2)a_n x^{n-3} + \cdots + (3 \cdot 2)a_3)|_{x=0} = (3 \cdot 2)a_3$$

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$$\begin{aligned} p(0) &= a_0 \\ p'(0) = (na_n x^{n-1} + \cdots + 3a_3x^2 + 2a_2x + a_1)|_{x=0} &= a_1 \\ p''(0) = (n(n-1)a_n x^{n-2} + \cdots + (3 \cdot 2)a_3 x + 2a_2)|_{x=0} &= 2a_2 \\ p'''(0) = (n(n-1)(n-2)a_n x^{n-3} + \cdots + (3 \cdot 2)a_3)|_{x=0} &= (3 \cdot 2)a_3 \end{aligned}$$
- In general, the kth derivative of p at 0 is $k!a_k$, up until the $(n+1)$st derivative, which is 0. So the Taylor series is:
$$(a_0) + (a_1)x + \frac{2!a_2}{2!}x^2 + \cdots + \frac{n!a_n}{n!}x^n + 0 + 0 + \cdots .$$

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$$(a_0) + (a_1)x + \frac{2!a_2}{2!}x^2 + \cdots + \frac{n!a_n}{n!}x^n + 0 + 0 + \cdots .$$
- Therefore, the Taylor series is just $p(x)$ itself! Should have expected this since cutting off the Taylor series at degree k gives the degree-k polynomial approximating the function.

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$$\log(1+x)\,|_{x=0} = \log(1) = 0$$
$$\frac{d}{dx}\log(1+x)\,|_{x=0} = \frac{1}{1+x}\,|_{x=0} = 1$$
$$\frac{d^2}{dx^2}\log(1+x)\,|_{x=0} = \frac{-1}{(1+x)^2}\,|_{x=0} = -1$$
$$\frac{d^3}{dx^3}\log(1+x)\,|_{x=0} = \frac{2}{(1+x)^3}\,|_{x=0} = 2$$
$$\frac{d^4}{dx^4}\log(1+x)\,|_{x=0} = \frac{-3!}{(1+x)^4}\,|_{x=0} = -3!$$
$$\frac{d^5}{dx^5}\log(1+x)\,|_{x=0} = \frac{4!}{(1+x)^5}\,|_{x=0} = 4!$$

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- Derivatives: $0, 1, -1, 2, -3!, 4!, \ldots$

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- Taylor series:

$$\log(1+x) = 0 + 1x - \frac{1!}{2!}x^2 + \frac{2!}{3!}x^3 - \frac{3!}{4!}x^4 + \frac{4!}{5!}x^5 - \cdots$$
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- This does not converge! Terms $2^n/n$ get really big, so even worse than $1 - 1 + 1 - 1 + \cdots$.
- Converges if $-1 < x \leq 1$.

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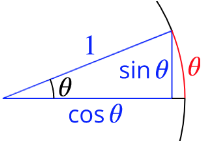
- Why are these true?



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&= \lim_{h\to 0} \frac{\cos x \sin h}{h} - \lim_{h\to 0} \frac{\sin x - \sin x \cos h}{h}
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&= \cos x \cdot 1 - \sin x \cdot 0
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Taylor series: trig functions

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- We have $\frac{d}{dx} \sin x = \cos x$ and (proved similarly) $\frac{d}{dx} \cos x = -\sin x$.

Taylor series: trig functions

- We have $\frac{d}{dx}\sin x = \cos x$ and (proved similarly) $\frac{d}{dx}\cos x = -\sin x$.
- Working out the Taylor series (around 0) for $\sin x$:

$$\begin{aligned} \sin 0 = 0, &\quad \frac{d}{dx}\sin x|_{x=0} = \cos 0 = 1, \\ \frac{d^2}{dx^2}\sin x|_{x=0} = -\sin 0 = 0, &\quad \frac{d^3}{dx^3}\sin x|_{x=0} = -\cos 0 = -1, \\ \frac{d^4}{dx^4}\sin x|_{x=0} = \sin 0 = 0, &\quad \frac{d^5}{dx^5}\sin x|_{x=0} = \cos 0 = 1, \\ \vdots \quad &\quad \vdots \end{aligned}$$

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- So Taylor series:

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

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$$\cos x = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \cdots$$

Derivatives of Taylor series

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- Let's look at $\frac{d}{dx}\sin x = \cos x$ in Taylor Series:

$$\begin{aligned}\frac{d}{dx}\sin x &= \frac{d}{dx}\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right)\\ &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\\ &= \cos x.\end{aligned}$$

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- If we substitute $-y = x$, then we get: $\frac{1}{1-y} = 1 + y + y^2 + \ldots$.

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- So:

$$e^{\log r + i\theta} = e^{\log r} e^{i\theta} = r(\cos\theta + i\sin\theta).$$

- With this definition, complex number multiplication becomes obvious:

$$e^{\log r_1 + i\theta_1} e^{\log r_2 + i\theta_2} = e^{(\log r_1 + \log r_2) + i(\theta_1 + \theta_2)} = e^{\log(r_1 r_2) + i(\theta_1 + \theta_2)}$$

immediately proves

$$\begin{aligned} & r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2 \left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right). \end{aligned}$$

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- "Most beautiful formula in math"
- Often written $e^{i\pi} + 1 = 0$, so includes arguably the five most important numbers in math.

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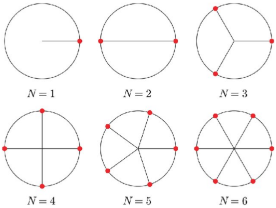
- Reminder: nth roots of unity are the solutions to $x^n - 1 = 0$.

$$x = \cos(2\pi k/n) + i\sin(2\pi k/n), \text{ for } k = 0, 1, \ldots, (n-1).$$



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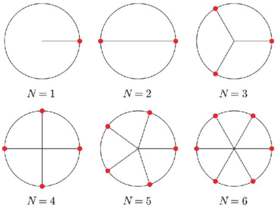

- In exponential form:

$$1, e^{2\pi i/n}, e^{4\pi i/n}, \ldots, e^{2(n-1)\pi i/n},$$

(note that $1 = e^{0\pi i/n}$)

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$$= \lim_{x \to 0} \frac{1}{1}.$$

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&= \lim_{x\to 0}\frac{f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots}{g'(0)x+\frac{g''(0)}{2!}x^2+\frac{g'''(0)}{3!}x^3+\cdots} \\
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&= \lim_{x\to 0}\frac{f'(0)}{g'(0)}
\end{aligned}$$

- This is called *L'Hôpital's Rule*: If $f(0) = g(0) = 0$, then

$$\lim_{x\to 0}(f(x)/g(x)) = f'(0)/g'(0).$$

Next time!

Probability I