COMP 761: Lecture 17 – Probability I

David Rolnick

October 14, 2020

Problem

A deck of n cards is shuffled randomly. How many cards are expected to be in the same place as they were before it was shuffled?

(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)

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- Office hours - tomorrow @ 10:30 am (Vincent), Friday @ 10 am (David)

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- A random variable can have *discrete values* (like the coin flips) or *continuous values* (like the height of a person).
- An *event* describes a particular set of outcomes for a random variable.
- Example: If X is the random variable for number of heads in 10 coin flips, then $(X = 0)$ is an event and $(X \geq 5)$ is an event.

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- Probabilities of mutually exclusive events sum together, e.g.

$$p(X = a \text{ or } X = b) = p(X = a) + p(X = b).$$

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- Intuitively, two events are independent if knowing the outcome of one does not affect the probability of the other.

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- For example, the expected number of heads for a single coin flip:

$$p(X = 0) \cdot 0 + p(X = 1) \cdot 1 = (1/2) \cdot 0 + (1/2) \cdot 1 = 1/2.$$

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- For example:

$$\mathbb{E}[\text{sum of two dice}] = \mathbb{E}[\text{one of the dice}] + \mathbb{E}[\text{the other one}].$$

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- The expected number of heads for a single flip is $1/2$.
- Therefore, by linearity of expectation, the expected total number is $n \cdot (1/2) = n/2$.

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- What is $\mathbb{E}[X_k]$?
- Each $\mathbb{E}[X_k] = 1/n$, so $\mathbb{E}[X] = n(1/n) = 1$.

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&= 1 + p + p^2 + \cdots \\
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Next time!

Probability II