COMP 761: Lecture 18 – Probability II

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Problem

A test for a disease is 90% accurate both for positives and negatives, meaning that 90% of the time a person with the disease tests positive, and 90% of the time a person without the disease tests negative. If 1% of the population has the disease, what is the chance that a randomly selected person who tests positive actually has the disease?

(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)

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- Problem 3: Simplifying the problem to save you some algebra – just set A equal to the identity matrix, so the problem is now $\frac{\partial}{\partial X}\,\mathrm{tr}(XBX)$.
- Clarification: During class, no need to read other chat responses, I realize they come in fast. I'll summarize any you need to read.

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- $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ always (linearity of expectation)
- $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$ *if* $X = x$ *and* $Y = y$ *are independent events*

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- Corollary of this other definition: $\text{Var}[X] \geq 0$ since it's the expected value of $(X - \mathbb{E}[X])^2$, which is always nonnegative.

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- We have $\mathbb{E}[X^2] = {}^1\!/_2 \cdot 0^2 + {}^1\!/_2 \cdot 1^2 = 1/2$.
- For the second term, we have $\mathbb{E}[X] = 1/2$.
- Therefore:

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 1/2 - (1/2)^2 = 1/4.$$

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- We have $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.
- We know $\mathbb{E}[(cX)^2] = c^2\mathbb{E}[X^2]$ and $\mathbb{E}[cX]^2 = c^2\mathbb{E}[X]^2$.

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- Therefore, $\mathrm{Var}[cX] = c^2 \mathrm{Var}[X]$.
- *Standard deviation* is defined as the square root of variance.
- So standard deviation of cX is c times the standard deviation of X.

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and we know the last term is 0 if $X = x$ and $Y = y$ are independent.

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- (So standard deviation is $\sqrt{n}/2$.)

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- What is $p(A \mid A)$?
- It is just $p(A \cap A)/p(A) = p(A)/p(A) = 1$.

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- Formally:

$$p(A \mid B) = \frac{p(A \cap B)}{p(B)} = \frac{1/8}{1/2} = 1/4.$$

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- What about if B is getting tails on the first flip?
- Then, $p(A \cap B) = 0$, so $p(A \mid B) = 0$.

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- What is $p(A \mid B) + p(A \mid \bar{B})$?
- It is *not* equal to $p(A)$:

$$
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p(A \mid B) + p(A \mid \bar{B}) &= \frac{p(A \cap B)}{p(B)} + \frac{p(A \cap \bar{B})}{p(\bar{B})} \\
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- The numerators add to $p(A)$ but different denominators.
- Instead, we have $p(A \mid B)p(B) + p(A \mid \bar{B})(1 - p(B)) = p(A)$.

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- This is called *Bayes' theorem*.

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- Definitely not! The second is in general greater than 1, so can't be a probability.

Problem

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$$p(B) = p(B \mid A)p(A) + p(B \mid \bar{A})p(\bar{A})$$

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- What is $p(B \mid \bar{A})$?
- We are given that $p(\bar{B} \mid \bar{A}) = 0.9$, so $p(B \mid \bar{A}) = 1 - 0.9 = 0.1$.

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- We know $p(A) = 0.01$ and $p(B \mid A) = 0.9$.
- $p(\bar{A}) = 1 - 0.01 = 0.99$.
- What is $p(B \mid \bar{A})$?
- We are given that $p(\bar{B} \mid \bar{A}) = 0.9$, so $p(B \mid \bar{A}) = 1 - 0.9 = 0.1$.
- Therefore:

$$p(B) = (0.9)(0.01) + (0.1)(0.99) = 0.108.$$

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- Putting it all together, we have:

$$\begin{aligned} p(A \mid B) &= \frac{p(B \mid A) \cdot p(A)}{p(B)} \\ &= \frac{(0.9)(0.01)}{0.108} \\ &\approx 0.0833. \end{aligned}$$

Next time!

Probability III