COMP 761: Lecture 21 – Linear Programs II

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October 23, 2020

Problem

Let P be the primal problem $\max_{x \in \mathbb{R}^n}(c \cdot x \mid Ax \leq b, \ x \geq 0)$ and let D be the dual problem $\min_{y \in \mathbb{R}^m}(b \cdot y \mid A^T y \geq c, \ y \geq 0)$. Then, if x is a feasible point for P and y is a feasible point for D, prove that $c \cdot x \leq b \cdot y$.

(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)

Course Announcements

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- No announcements!

Standard form

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- Reminder: the *standard form* of a LP has:
  1. The optimization as a max not a min
  2. All variables x_k constrained by $x_k \geq 0$
  3. All other inequality constraints as $\leq$ not $\geq$
  4. No equality constraints

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  1. The optimization as a max not a min
  2. All variables x_k constrained by $x_k \geq 0$
  3. All other inequality constraints as $\leq$ not $\geq$
  4. No equality constraints
- Last time: We can convert any LP into an equivalent LP in standard form.

The dual

The dual

- Suppose we have an LP in standard form, e.g.

$$\begin{aligned} \max_{x\in\mathbb{R}^3} \quad & 2x_1 + x_2 - x_3 \\ \text{such that} \quad & 2x_1 + x_3 \leq 4 \\ & x_2 - x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

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$$24 = 4 + 2(10) \geq (2x_1 + x_3) + 2(x_2 - x_3) = 2x_1 + 2x_2 - x_3.$$

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- We know $x_2 \geq 0$, so $2x_1 + 2x_2 - x_3 \geq 2x_1 + x_2 - x_3$.
- So the optimum is at most 24.

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- If the coefficient of x_i in constraint j is a_{ij}, and the coefficient of x_i in the objective is c_i, then we need

$$\sum_j a_{ij} y_j \geq c_i.$$

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- Then, our bound is $\sum_j b_j y_j$, where b_j are upper bounds in the inequalities on x (4 and 10 in this case).
- We want this as *small* as possible so it gives a good upper bound.

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- A general LP in standard form can be written as:

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where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

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- (You can look up how to get the dual if the LP *isn't* in standard form – it's also pretty easy to work out yourself.)

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- So the dual of the dual is:

$$\min_{x \in \mathbb{R}^n} (-c \cdot x) \text{ such that } -Ax \geq -b \text{ and } x \geq 0.$$

which is just equivalent to the primal!

Weak duality

Weak Duality Theorem. Let P be the primal problem $\max_{x \in \mathbb{R}^n}(c \cdot x \mid Ax \leq b, x \geq 0)$ and let D be the dual problem $\min_{y \in \mathbb{R}^m}(b \cdot y \mid A^T y \geq c, y \geq 0)$. Then, if x is a feasible point for P and y is a feasible point for D, we have $c \cdot x \leq b \cdot y$.

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- We know that:

$$c \cdot x = x^T c \leq x^T (A^T y).$$

  since $x \geq 0$ and $A^T y \geq c$.

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- Putting it all together, we have

$$c \cdot x \leq x^T (A^T y) \leq b^T y = b \cdot y.$$

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- Do we immediately know that $c \cdot x^* = b \cdot y^*$?
- In fact, this is true, but this theorem isn't enough to prove it.
- Let's try anyway and see where it fails.
- If we write D as $\max(-b \cdot y)$ and P as $\min(-c \cdot x)$, then we can apply the theorem again since P is the dual of D:

$$-b \cdot y^* \leq -c \cdot x^*,$$

...which is the same as $c \cdot x^* \leq b \cdot y^*$ and doesn't help.

Strong duality

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- We will not prove this here.
- Possibilities for P and D:
  1. P is unbounded and D is infeasible.
  2. D is unbounded and P is infeasible.
  3. P and D are both infeasible.
  4. Both are feasible and bounded, and they have the same optimum (a max for the primal, a min for the dual).

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- Specific LPs may be much easier to solve in dual form.
- Dual provides a “certificate” of optimality.

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- We can introduce *slack variables* to make $\leq$ constraints into $=$

$$\begin{aligned} \max_{x \in \mathbb{R}^5} \quad & 2x_1 + x_2 - x_3 \\ \text{such that} \quad & 2x_1 + x_3 + x_4 = 4 \\ & x_1 + x_2 - x_3 + x_5 = 10 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

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\max_{x \in \mathbb{R}^5} \quad & 2x_1 + x_2 - x_3 \\
\text{such that} \quad & x_4 = 4 - 2x_1 - x_3 \\
& x_5 = 10 - x_1 - x_2 + x_3 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{aligned}
$$

Intuition for simplex method

$$\begin{aligned} \max_{x \in \mathbb{R}^5} \quad & 2x_1 + x_2 - x_3 \\ \text{such that} \quad & x_4 = 4 - 2x_1 - x_3 \\ & x_5 = 10 - x_1 - x_2 + x_3 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

- Is this LP feasible?

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& x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{aligned}$$

- There is an obvious feasible point, just setting $x_4 = 4$ and $x_5 = 10$, with $x_1 = x_2 = x_3 = 0$.

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- There is an obvious feasible point, just setting $x_4 = 4$ and $x_5 = 10$, with $x_1 = x_2 = x_3 = 0$.
- What is the objective value at this point?

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- There is an obvious feasible point, just setting $x_4 = 4$ and $x_5 = 10$, with $x_1 = x_2 = x_3 = 0$.
- The objective value here is just 0.

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- There is an obvious feasible point, just setting $x_4 = 4$ and $x_5 = 10$, with $x_1 = x_2 = x_3 = 0$.
- The objective value here is just 0.
- Let's try improving the objective, e.g. by increasing x_1.

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- There is an obvious feasible point, just setting $x_4 = 4$ and $x_5 = 10$, with $x_1 = x_2 = x_3 = 0$.
- The objective value here is just 0.
- Let's try improving the objective, e.g. by increasing x_1.
- Can we just increase x_1 as much as we want?

Intuition for simplex method

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- There is an obvious feasible point, just setting $x_4 = 4$ and $x_5 = 10$, with $x_1 = x_2 = x_3 = 0$.
- The objective value here is just 0.
- Let's try improving the objective, e.g. by increasing x_1.
- Increasing x_1 from 0 to c means we need to decrease x_4 and x_5.

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- There is an obvious feasible point, just setting $x_4 = 4$ and $x_5 = 10$, with $x_1 = x_2 = x_3 = 0$.
- The objective value here is just 0.
- Let's try improving the objective, e.g. by increasing x_1.
- Increasing x_1 from 0 to c means we need to decrease x_4 and x_5.
- We start with $x_4 = 4$ and $x_5 = 10$ and get $x_4 = 4 - 2c$ and $x_5 = 10 - c$.

Intuition for simplex method

$$\begin{aligned} \max_{x \in \mathbb{R}^5} \quad & 2x_1 + x_2 - x_3 \\ \text{such that} \quad & x_4 = 4 - 2x_1 - x_3 \\ & x_5 = 10 - x_1 - x_2 + x_3 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

- There is an obvious feasible point, just setting $x_4 = 4$ and $x_5 = 10$, with $x_1 = x_2 = x_3 = 0$.
- The objective value here is just 0.
- Let's try improving the objective, e.g. by increasing x_1.
- Increasing x_1 from 0 to c means we need to decrease x_4 and x_5.
- We start with $x_4 = 4$ and $x_5 = 10$ and get $x_4 = 4 - 2c$ and $x_5 = 10 - c$.
- Since $x_4, x_5 \geq 0$ we have $c \leq 2$.

Intuition for simplex method

$$\begin{aligned} \max_{x \in \mathbb{R}^5} \quad & 2x_1 + x_2 - x_3 \\ \text{such that} \quad & x_4 = 4 - 2x_1 - x_3 \\ & x_5 = 10 - x_1 - x_2 + x_3 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

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- Let's try improving the objective, e.g. by increasing x_1.
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- We start with $x_4 = 4$ and $x_5 = 10$ and get $x_4 = 4 - 2c$ and $x_5 = 10 - c$.
- Since $x_4, x_5 \geq 0$ we have $c \leq 2$.
- Let's set x_1 to 2, so now $x_4 = 0$.

Intuition for simplex method

$$\begin{aligned}
\max_{x \in \mathbb{R}^5} \quad & 2x_1 + x_2 - x_3 \\
\text{such that} \quad & x_4 = 4 - 2x_1 - x_3 \\
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- There is an obvious feasible point, just setting $x_4 = 4$ and $x_5 = 10$, with $x_1 = x_2 = x_3 = 0$.
- The objective value here is just 0.
- Let's try improving the objective, e.g. by increasing x_1.
- Increasing x_1 from 0 to c means we need to decrease x_4 and x_5.
- We start with $x_4 = 4$ and $x_5 = 10$ and get $x_4 = 4 - 2c$ and $x_5 = 10 - c$.
- Since $x_4, x_5 \geq 0$ we have $c \leq 2$.
- Let's set x_1 to 2, so now $x_4 = 0$.
- We can rearrange the set of equations above.

Intuition for simplex method

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& x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{aligned}
$$

Intuition for simplex method

$$\max_{x \in \mathbb{R}^5} \quad 2x_1 + x_2 - x_3$$

$$\text{such that} \quad x_1 = 2 - x_3/2 - x_4/2$$

$$x_5 = 10 - x_1 - x_2 + x_3$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0.$$

Intuition for simplex method

$$\begin{aligned} \max_{x \in \mathbb{R}^5} \quad & 2(2 - x_3/2 - x_4/2) + x_2 - x_3 \\ \text{such that} \quad & x_1 = 2 - x_3/2 - x_4/2 \\ & x_5 = 10 - (2 - x_3/2 - x_4/2) - x_2 + x_3 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

Intuition for simplex method

$$\begin{aligned} \max_{x \in \mathbb{R}^5} \quad & 4 + x_2 - 2x_3 - x_4 \\ \text{such that} \quad & x_1 = 2 - x_3/2 - x_4/2 \\ & x_5 = 8 - x_2 + 3x_3/2 + x_4/2 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

Intuition for simplex method

$$\max_{x \in \mathbb{R}^5} \quad 4 + x_2 - 2x_3 - x_4$$
$$\text{such that} \quad x_1 = 2 - x_3/2 - x_4/2$$
$$x_5 = 8 - x_2 + 3x_3/2 + x_4/2$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0.$$

- Now our provisional solution is $x_1 = 2$, $x_5 = 8$, and $x_2 = x_3 = x_4 = 0$.

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- Now our provisional solution is $x_1 = 2$, $x_5 = 8$, and $x_2 = x_3 = x_4 = 0$.
- The objective is now 4, better than 0 before.

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- Now our provisional solution is $x_1 = 2$, $x_5 = 8$, and $x_2 = x_3 = x_4 = 0$.
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- What we just did is called a *pivot* with *entering variable* x_1 and *leaving variable* x_4.

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- Now our provisional solution is $x_1 = 2$, $x_5 = 8$, and $x_2 = x_3 = x_4 = 0$.
- The objective is now 4, better than 0 before.
- What we just did is called a *pivot* with *entering variable* x_1 and *leaving variable* x_4.
- Let's pivot again!

Intuition for simplex method

$$\max_{x \in \mathbb{R}^5} \quad 4 + x_2 - 2x_3 - x_4$$
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$$x_5 = 8 - x_2 + 3x_3/2 + x_4/2$$
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- Now our provisional solution is $x_1 = 2$, $x_5 = 8$, and $x_2 = x_3 = x_4 = 0$.
- The objective is now 4, better than 0 before.
- What we just did is called a *pivot* with *entering variable* x_1 and *leaving variable* x_4.
- Let's pivot again!
- What should be the entering variable?

Intuition for simplex method

$$\begin{aligned}
\max_{x \in \mathbb{R}^5} \quad & 4 + x_2 - 2x_3 - x_4 \\
\text{such that} \quad & x_1 = 2 - x_3/2 - x_4/2 \\
& x_5 = 8 - x_2 + 3x_3/2 + x_4/2 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0.
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- Now our provisional solution is $x_1 = 2$, $x_5 = 8$, and $x_2 = x_3 = x_4 = 0$.
- The objective is now 4, better than 0 before.
- What we just did is called a *pivot* with *entering variable* x_1 and *leaving variable* x_4.
- Let's pivot again!
- The entering variable must have coefficient > 0 in the objective.

Intuition for simplex method

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- Now our provisional solution is $x_1 = 2$, $x_5 = 8$, and $x_2 = x_3 = x_4 = 0$.
- The objective is now 4, better than 0 before.
- What we just did is called a *pivot* with *entering variable* x_1 and *leaving variable* x_4.
- Let's pivot again!
- The entering variable must have coefficient > 0 in the objective.
- So we pick x_2.

Intuition for simplex method

Intuition for simplex method

$$\begin{aligned}
\max_{x \in \mathbb{R}^5} \quad & 4 + x_2 - 2x_3 - x_4 \\
\text{such that} \quad & x_1 = 2 - x_3/2 - x_4/2 \\
& x_2 = 8 + 3x_3/2 + x_4/2 - x_5 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{aligned}$$

Intuition for simplex method

$$\begin{aligned}
\max_{x\in\mathbb{R}^5} \quad & 4 + (8 + 3x_3/2 + x_4/2 - x_5) - 2x_3 - x_4 \\
\text{such that} \quad & x_1 = 2 - x_3/2 - x_4/2 \\
& x_2 = 8 + 3x_3/2 + x_4/2 - x_5 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{aligned}$$

Intuition for simplex method

$$\begin{aligned}\max_{x\in\mathbb{R}^5} \quad & 12 - x_3/2 - x_4/2 - x_5 \\ \text{such that} \quad & x_1 = 2 - x_3/2 - x_4/2 \\ & x_2 = 8 + 3x_3/2 + x_4/2 - x_5 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0.\end{aligned}$$

Intuition for simplex method

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- Our provisional solution is $x_1 = 2$, $x_2 = 8$, and $x_3 = x_4 = x_5 = 0$.

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- Our provisional solution is $x_1 = 2$, $x_2 = 8$, and $x_3 = x_4 = x_5 = 0$.
- Can we make it any better?

Intuition for simplex method

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- Our provisional solution is $x_1 = 2$, $x_2 = 8$, and $x_3 = x_4 = x_5 = 0$.
- Can we make it any better?
- No! All variables in the objective have negative coefficients, but the variables are ≥ 0.

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- Our provisional solution is $x_1 = 2$, $x_2 = 8$, and $x_3 = x_4 = x_5 = 0$.
- Can we make it any better?
- No! All variables in the objective have negative coefficients, but the variables are ≥ 0.
- So we are done, and this is the maximum value for the objective.

Simplex method

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- Use slack variables to rewrite the LP.

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- Choose an entering variable with positive coefficient in the objective.
- Choose the leaving variable as the first variable to decrease to 0 as we increase the entering variable.
- Rewrite equations in terms of the new RHS variables.
- Repeat until no more positive coefficients in objective.

Problem 1: Initialization

$$\begin{aligned} \max_{x \in \mathbb{R}^5} \quad & 2x_1 + x_2 - x_3 \\ \text{such that} \quad & x_4 = 4 - 2x_1 - x_3 \\ & x_5 = 10 - x_1 - x_2 + x_3 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{aligned}$$

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- We assumed that we have a feasible solution to start with.
- What could go wrong?

Problem 1: Initialization

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- We assumed that we have a feasible solution to start with.
- Some of the constants could be negative.

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- We need a way to reorganize the LP so we can start at a feasible solution.

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- We assumed that we have a feasible solution to start with.
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- We need a way to reorganize the LP so we can start at a feasible solution.
- Will skip details, but outline: Weaken all the constraints so there is a feasible solution, then gradually tighten them.
- Can use the simplex method itself in order to tighten them fully, or else work out that the problem is infeasible.

Problem 2: Pivot rules

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- So might keep on pivoting forever without reaching the optimum.
- Can prove that with a good rule for which entering variables to choose (e.g. picking candidates x_k with the smallest k), cannot go on forever.

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- In this case, the LP does not have an optimum – it's unbounded.

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- Consider the polytope of feasible points.
- Recall a vertex is a point with $\#$ tight constraints equal to the dimension (here that is the $\#$ of variables, $m + n$).

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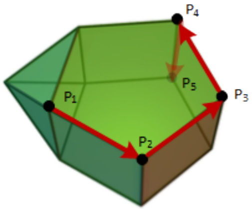

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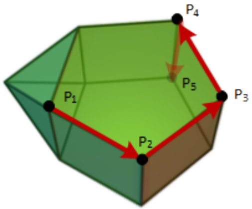

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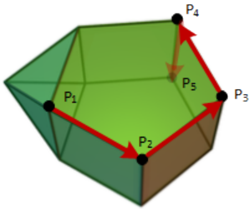

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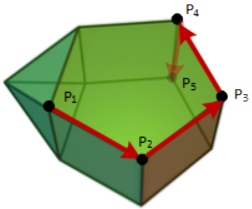

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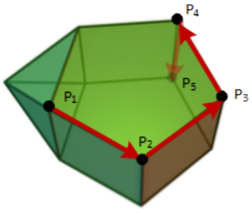

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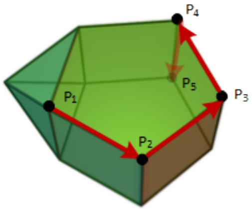

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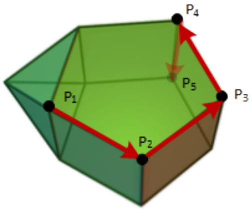

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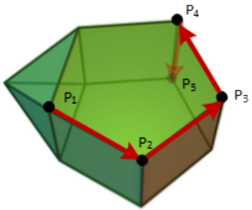
- This corresponds to a neighboring vertex.
- Eventually we arrive at the optimal vertex.

Other methods for solving LPs



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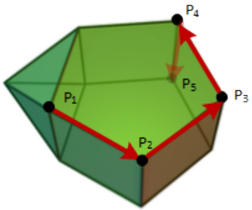

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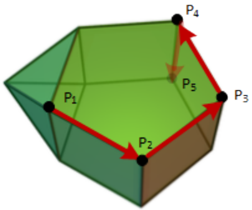

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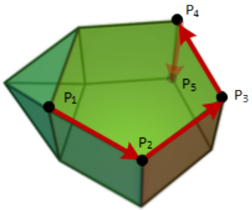

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- By contrast, *interior-point methods* (e.g. Karmarkar's algorithm) stay inside the polytope until the last moment.
- Ellipsoid method – convert to a problem of just finding a feasible point and pick gradually smaller ellipsoids containing the polytope.

Next time!

Graph Algorithms I