COMP 761: Lecture 22 – Sorting

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Problem

Prove that for integers $n \geq 2$, we have:

$$\frac{1}{2} + \cdots + \frac{1}{n} \leq \log n \leq 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}.$$

(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)

Course Announcements

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- Office hours today right after class!

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- To prove this, note that it's true if for every $c > 0$, we have $x \log x < cx^2$ for x sufficiently large.
- And $x \log x < cx^2$ is equivalent to $(1/c) \log x < x$, which is equivalent to $x^{1/c} < e^x$ and is true for x large enough.

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- It isn't possible to have both of these be true simultaneously!

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- So for example $2x^{10} > (x+1)^{10}$ for x sufficiently large, so $(x+1)^{10} = O(x^{10})$.
- And $(x+1)^{10} = \Omega(x^{10})$ is clear, since $(x+1)^{10} > x^{10}$.

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- Same statements also hold for $o(\cdot)$, $\Omega(\cdot)$, $\omega(\cdot)$, and $\Theta(\cdot)$.

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- Or *average-case* time (average amount of time it takes to run over all possible inputs).

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- The overall time of the whole algorithm is then $m = \sum_{k=1}^{n} m_k$.
- What is the worst-case time complexity of the algorithm?
- In the worst case, $m_k = k$ for each k (the list starts out in reverse order), so we have

$$m = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \Theta(n^2).$$

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- In this case, both worst-case and average-case times are $\Theta(n^2)$.

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- Example:

$$\begin{aligned}
\text{sort}([5,2,6,1,4,3]) &= \text{merge}(\text{sort}([5,2,6]), \text{sort}([1,4,3])) \\
&= \text{merge}(\text{merge}(\text{sort}([5,2]), \text{sort}([6])), \\
&\quad\ \text{merge}(\text{sort}([1,4]), \text{sort}([3]))) \\
&= \text{merge}(\text{merge}([2,5],[6]), \text{merge}([1,4],[3])) \\
&= \text{merge}([2,5,6],[1,3,4]) \\
&= [1,2,3,4,5,6]
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- Worst case, we have to flip all cards, so $c(n_1 + n_2)$ for some c.

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- Can show that average-case time is also $\Theta(n \log n)$.

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Prove that for integers $n \geq 2$, we have:

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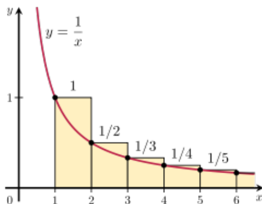

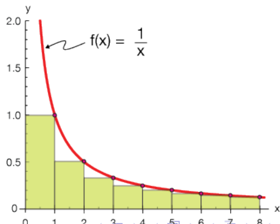

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- In this case, we have:

$$\begin{aligned} f(n) &= f(n-1) + f(1) + cn \\ &= f(n-1) + cn \\ &= cn + c(n-1) + \ldots + c \\ &= c(n + (n-1) + \ldots + 1) \\ &= cn(n-1)/2 \\ &= \Theta(n^2). \end{aligned}$$

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- By linearity of expectation:

$$\mathbb{E}[X] = \sum_{i<j} \mathbb{E}[X_{ij}] = \sum_{i<j} p(\text{comparing } z_i \text{ and } z_j)$$

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- Therefore, we have:

$$\begin{aligned} & p(\text{comparing } z_i \text{ and } z_j) \\ = \; & p(\text{picking } z_i \text{ first out of the nums between } z_i \text{ and } z_j) \\ & + p(\text{picking } z_j \text{ first out of the nums between } z_i \text{ and } z_j) \\ = \; & \frac{1}{j-i+1} + \frac{1}{j-i+1} \\ = \; & \frac{2}{j-i+1}. \end{aligned}$$

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- Where we used $\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} = O(\log n)$.
- Since best-case running time is $\Theta(n \log n)$, the average running time is not just $O(n \log n)$, it is $\Theta(n \log n)$.

Next time!

Heaps