COMP 761: Lecture 24 – Graph Algorithms I

David Rolnick

October 30, 2020

Problem

Find a way to remove the maximum from a heap of n elements, while preserving the heap property, in $O(\log n)$ time.

(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)

Course Announcements

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- For Problem 1 on the Problem Set, please do not just cite a result about n^k and $O(\cdot)$ or $\Omega(\cdot)$, would like at least an explanation for why the definitions of $O(\cdot)$ and $\Omega(\cdot)$ hold here.

Review: Heaps

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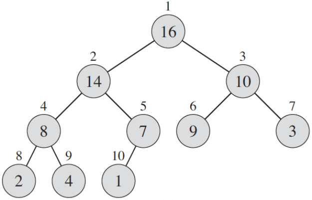
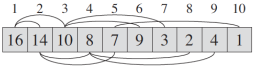
- The root node always must have the largest key.





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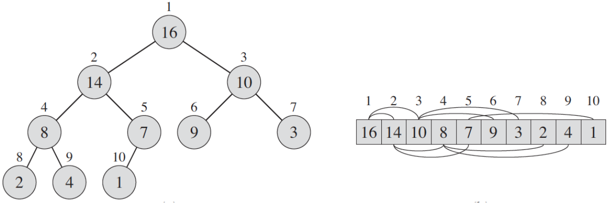

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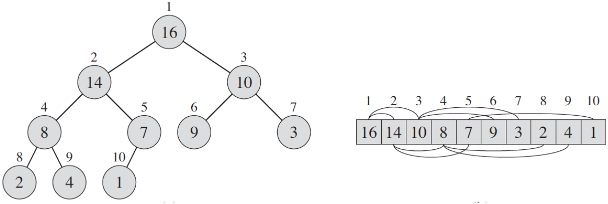

- We also assume in a heap that each *row* (set of nodes of a single depth) is full except possibly the last one, so depth = $\Theta(\log n)$.
- And that the last row has all its nodes as far left as possible (easy by swapping left and right).

Review: Priority queues

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- A *max-priority queue* is a data structure S that allows you to perform the following operations
  - Insert(S, x), inserting key x into S.
  - Maximum(S), returns the maximum of S.
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  - IncreaseKey(S, i, x) takes the element at index i and increases it to value x (assuming the key was smaller before).

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- Priority queues are useful across algorithms.
- A max-heap can be used to implement a max-priority queue!
- (Likewise, a min-heap can be used to implement a min-priority queue.)

Heaps as priority queues

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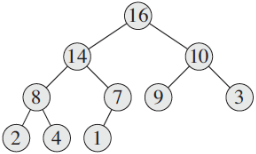
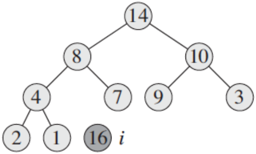
- Do we have any of these already in a max-heap?
- We already have Maximum, since it's the root.
- Let's now try to do ExtractMax.

ExtractMax

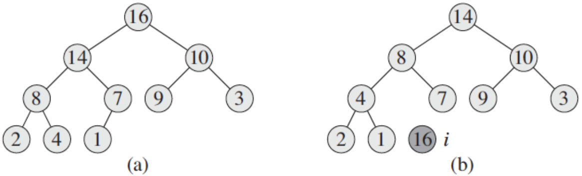


(a)



(b)

ExtractMax



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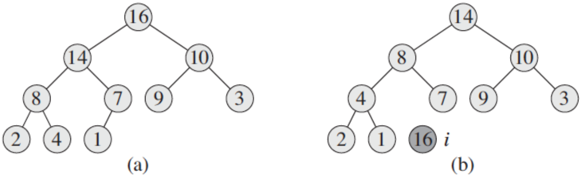

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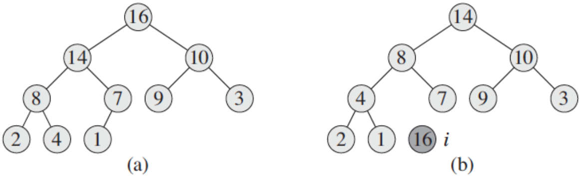

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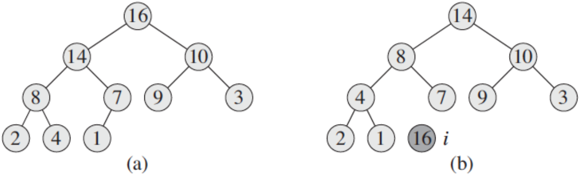

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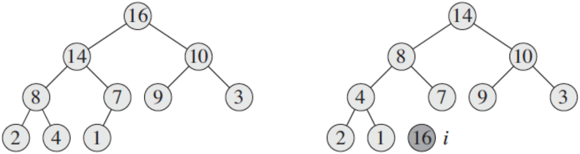

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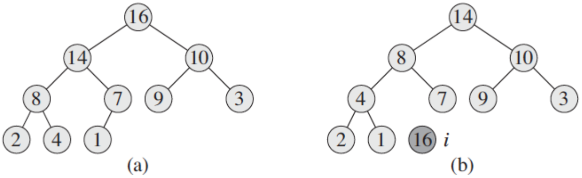

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- But the two sub-trees below the root must both be heaps.
- So we can do the iterative swapping thing we did before to make a heap.
- $\#$ swaps = depth of whole tree = $O(\log n)$.
- So ExtractMax runs in $O(\log n)$.

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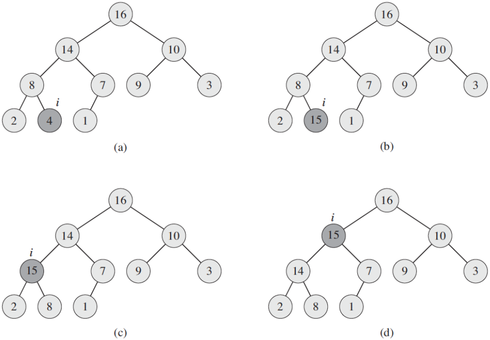

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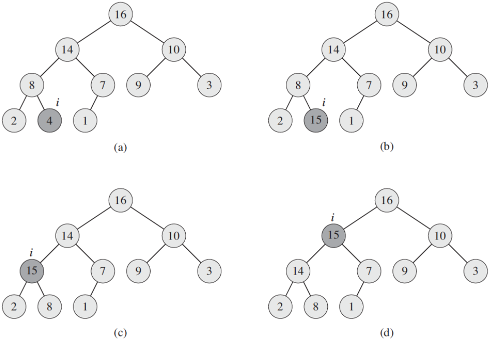

- This takes $O(\log n)$ moves.

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- The time is just the time for IncreaseKey, $O(\log n)$.

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- We have a way to use a max-heap as a max-priority queue.
- Each of the operations Insert, Maximum, ExtractMax, and IncreaseKey runs in $O(\log n)$ time (and Maximum is just $O(1)$ time).
- We'll see soon why this is useful.

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Algorithm	Worst-case running time	Average-case/expected running time
Insertion sort	$\Theta(n^2)$	$\Theta(n^2)$
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- Each ExtractMax call takes time $O(\log n)$.
- Total (worst-case) time $n \cdot O(\log n) = O(n \log n)$.

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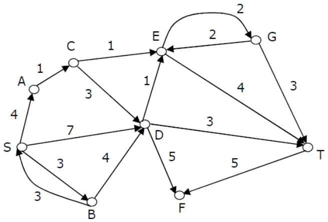

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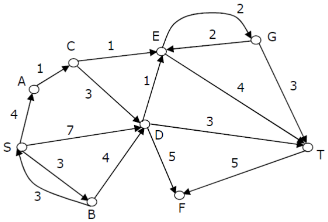

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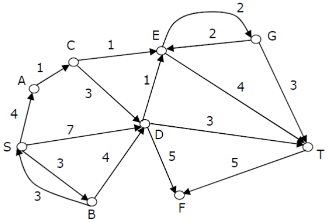

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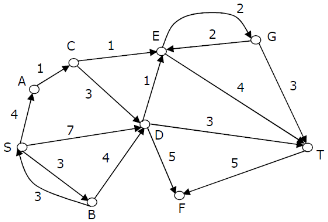

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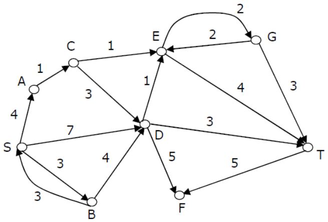

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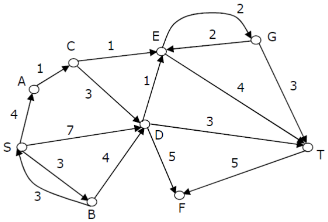
- If (i, j) and (j, i) both exist, then can have different weights on them.

Shortest paths



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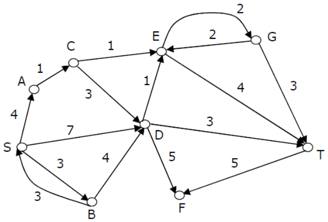

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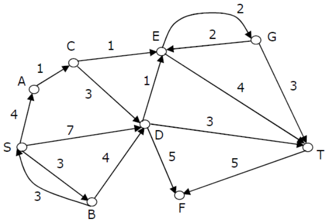

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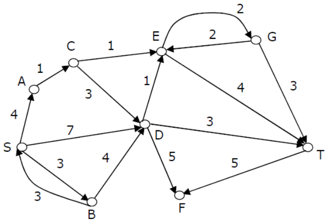

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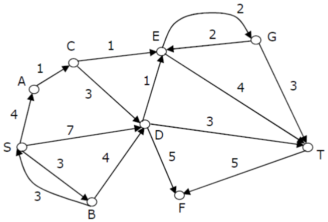

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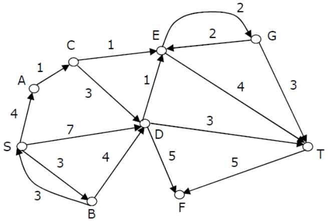
- A *shortest path* between *s* and *t* is a path with minimal length between them. (There might be several such paths).
- We can also do this with an unweighted graph (all weights = 1) or an undirected graph (edges go both ways).

Dijkstra's algorithm



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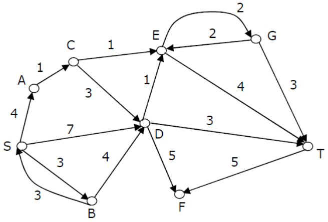

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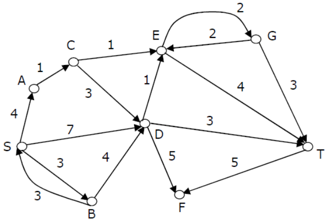

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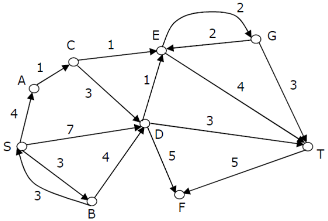

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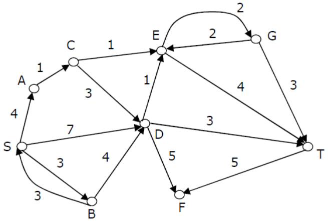

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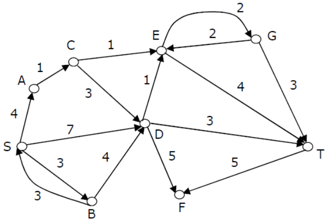

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- Similar logic, vertex $a = \text{argmin}_q(\min(w_{sq}, w_{sb} + w_{bq}))$.

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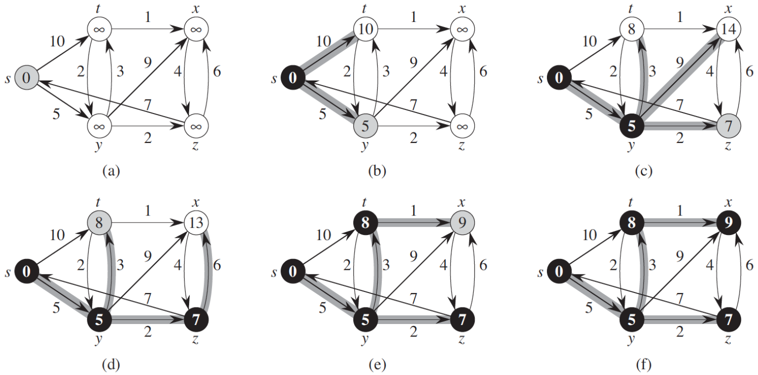

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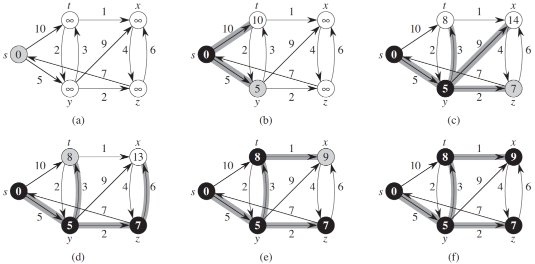

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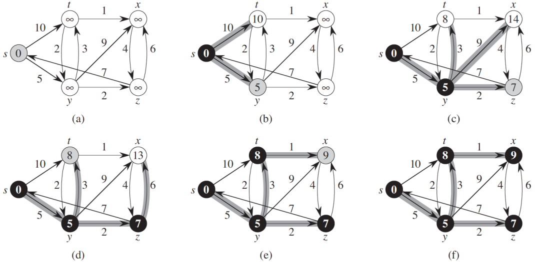

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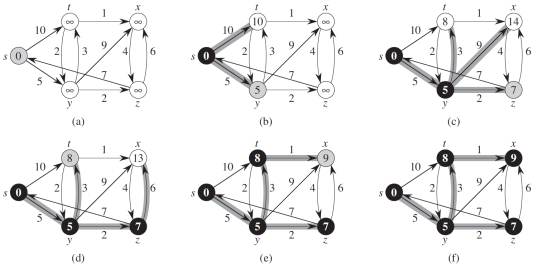

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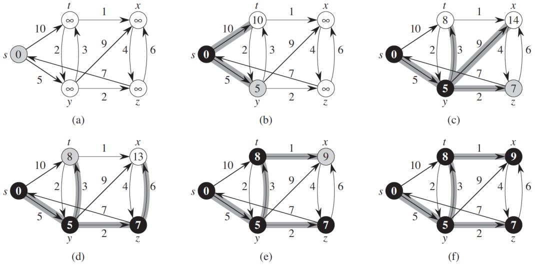

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- At each step, pick unvisited vertex *i* with smallest distance estimate (starting with *s*) - this estimate has to be correct.

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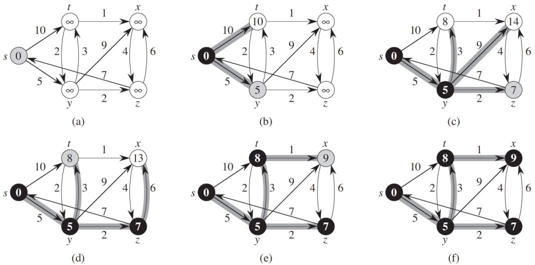

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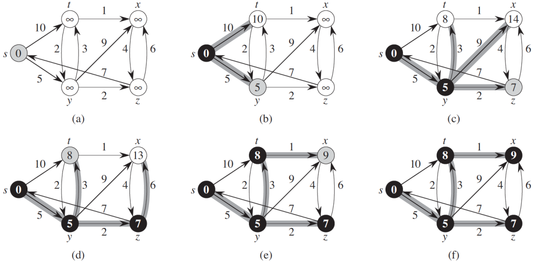
- Start out with ∞ everywhere, s at distance 0 from itself.
- At each step, pick unvisited vertex i with smallest distance estimate (starting with s) - this estimate has to be correct.
- Visit all its neighbors j, and update distance estimate $d(j)$ to $\min(d(j), d(i) + w_{ij})$.

Dijkstra's algorithm



- Maintain guesses for distances of all vertices from s.
- Start out with ∞ everywhere, s at distance 0 from itself.
- At each step, pick unvisited vertex i with smallest distance estimate (starting with s) - this estimate has to be correct.
- Visit all its neighbors j, and update distance estimate $d(j)$ to $\min(d(j), d(i) + w_{ij})$.
- Repeat.

Next time!

Graph algorithms II