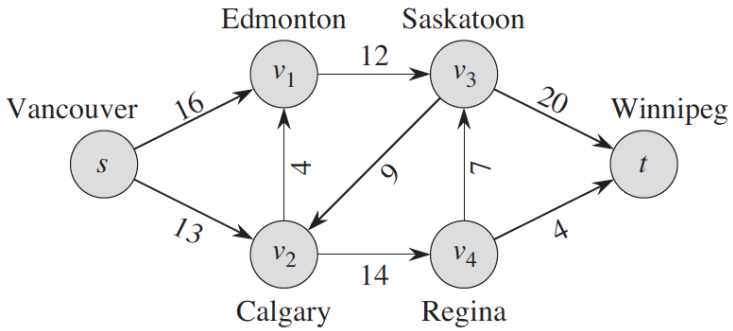
COMP 761: Lecture 26 – Graph Algorithms III

David Rolnick

November 4, 2020

Problem

A company would like to transport some goods from Vancouver to Winnipeg, but the maximum amount they can ship between any two cities en route is limited by the capacity of their trucks, as shown in the graph. How many of these items can they ship?



(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)

Course Announcements

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- Office hours: Vincent 10:30 am Thu, David 10:00 am Fri

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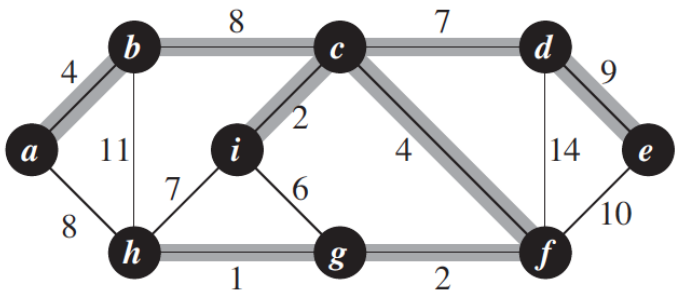

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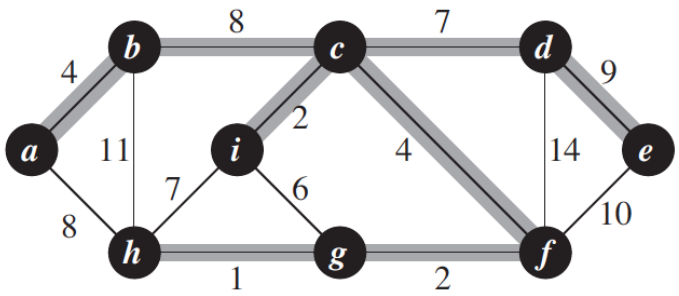

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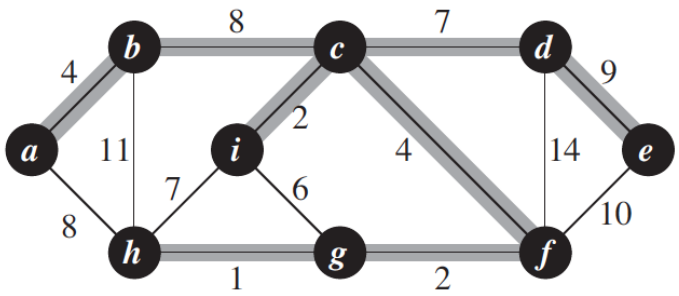

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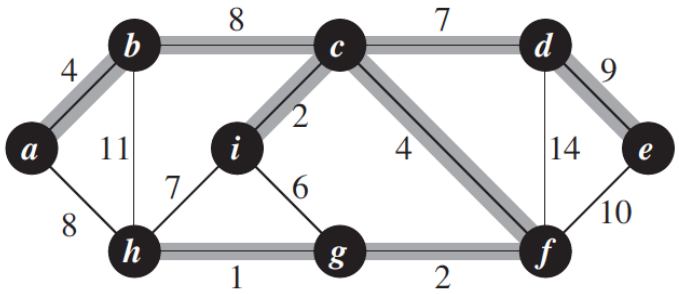

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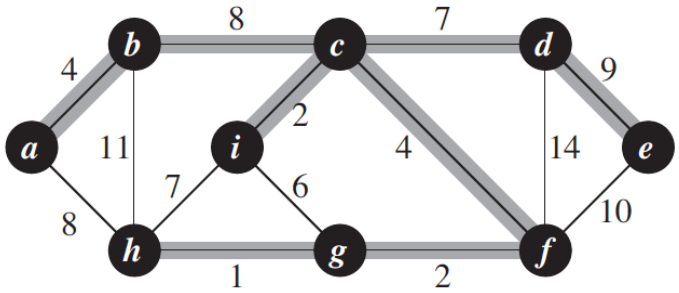

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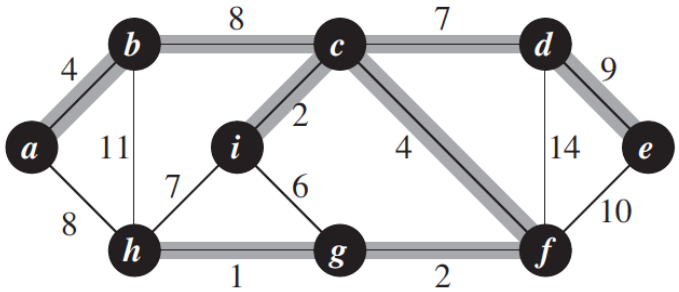

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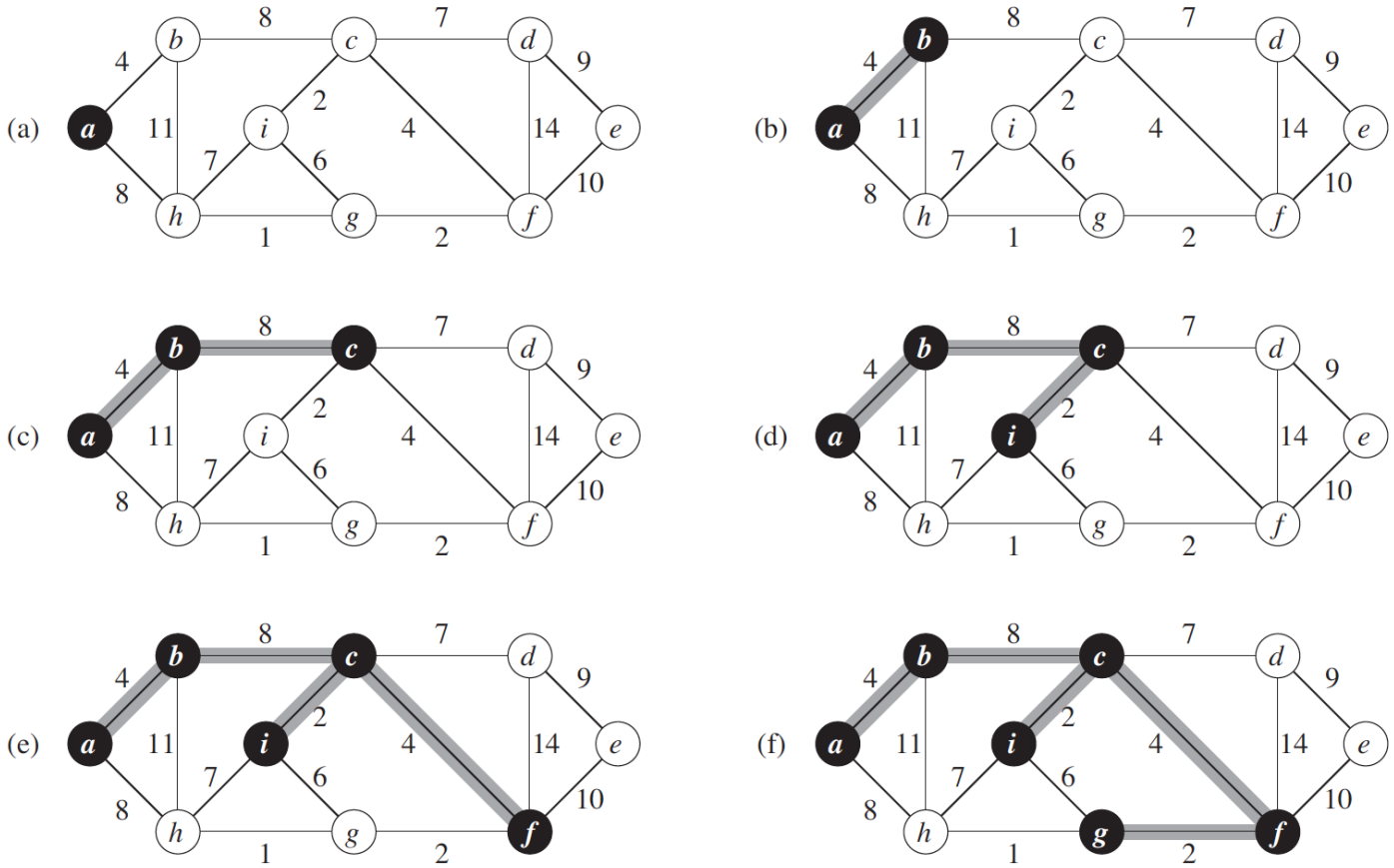
- Biological solution to a similar problem by slime molds:

Prim's algorithm



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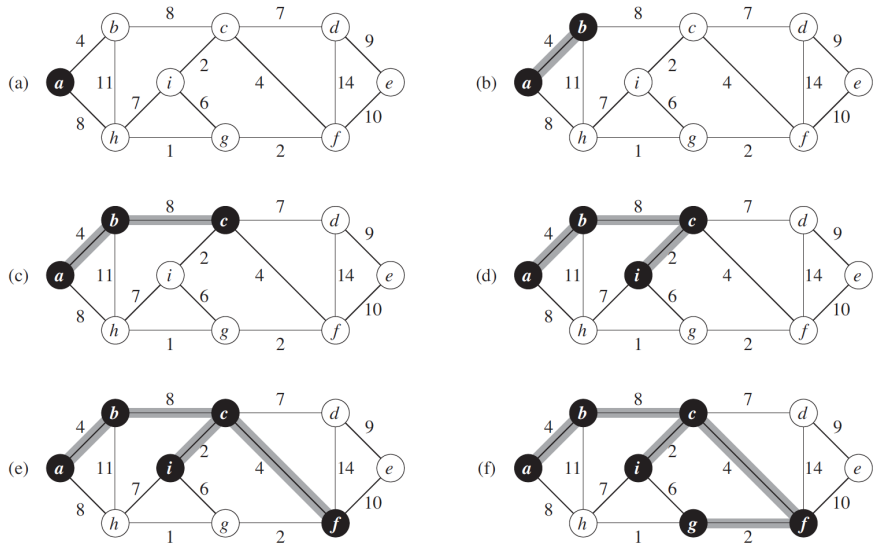

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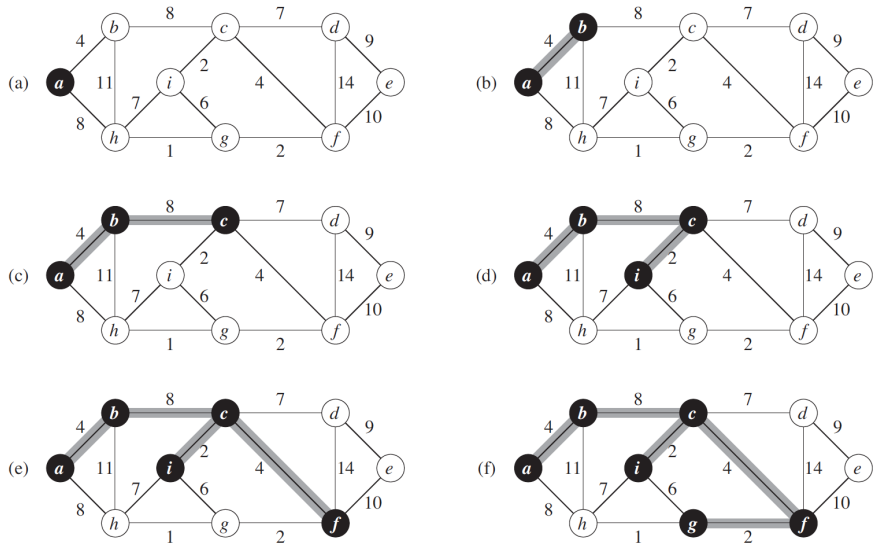

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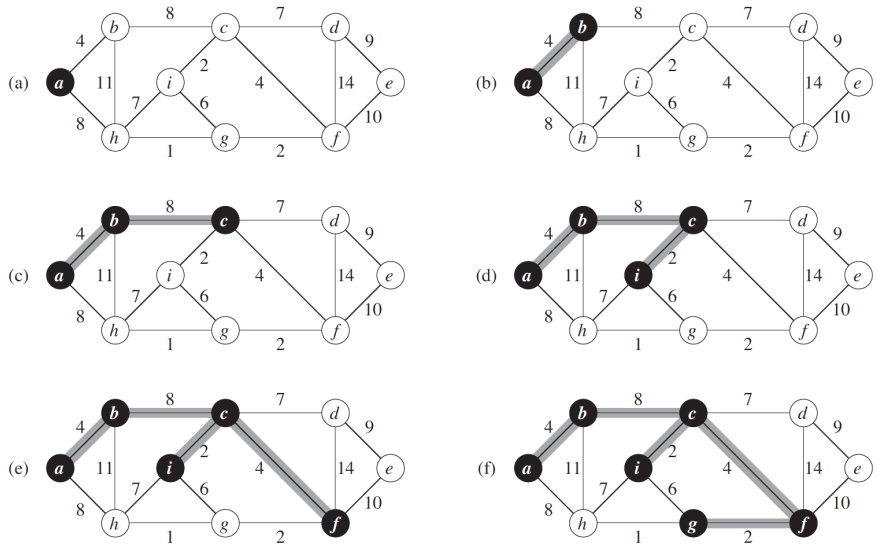

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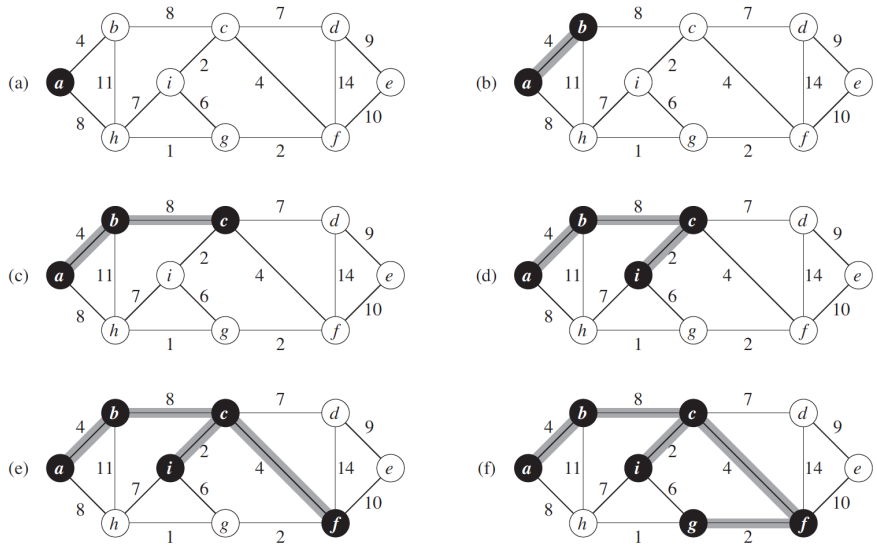

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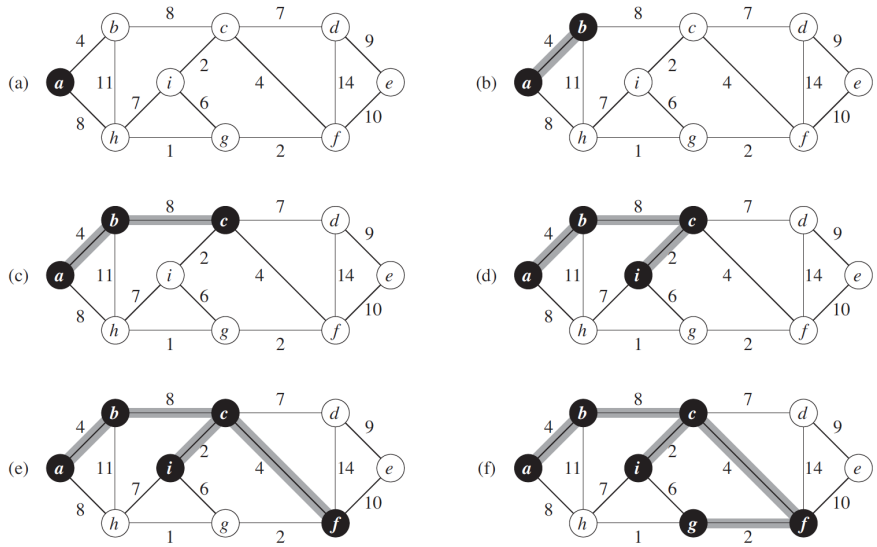

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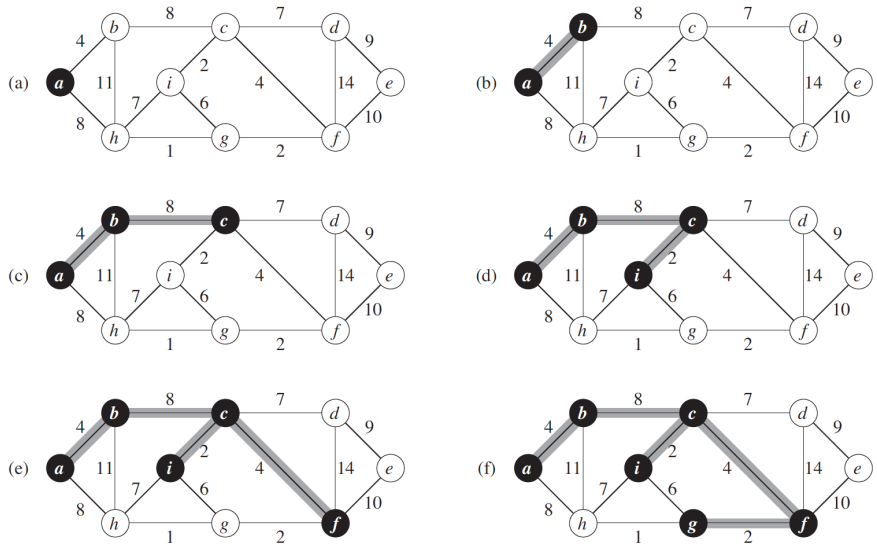

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- Why is it minimal weight? What proof technique can we use?

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- This is a contradiction, so e is in M and the inductive step holds.

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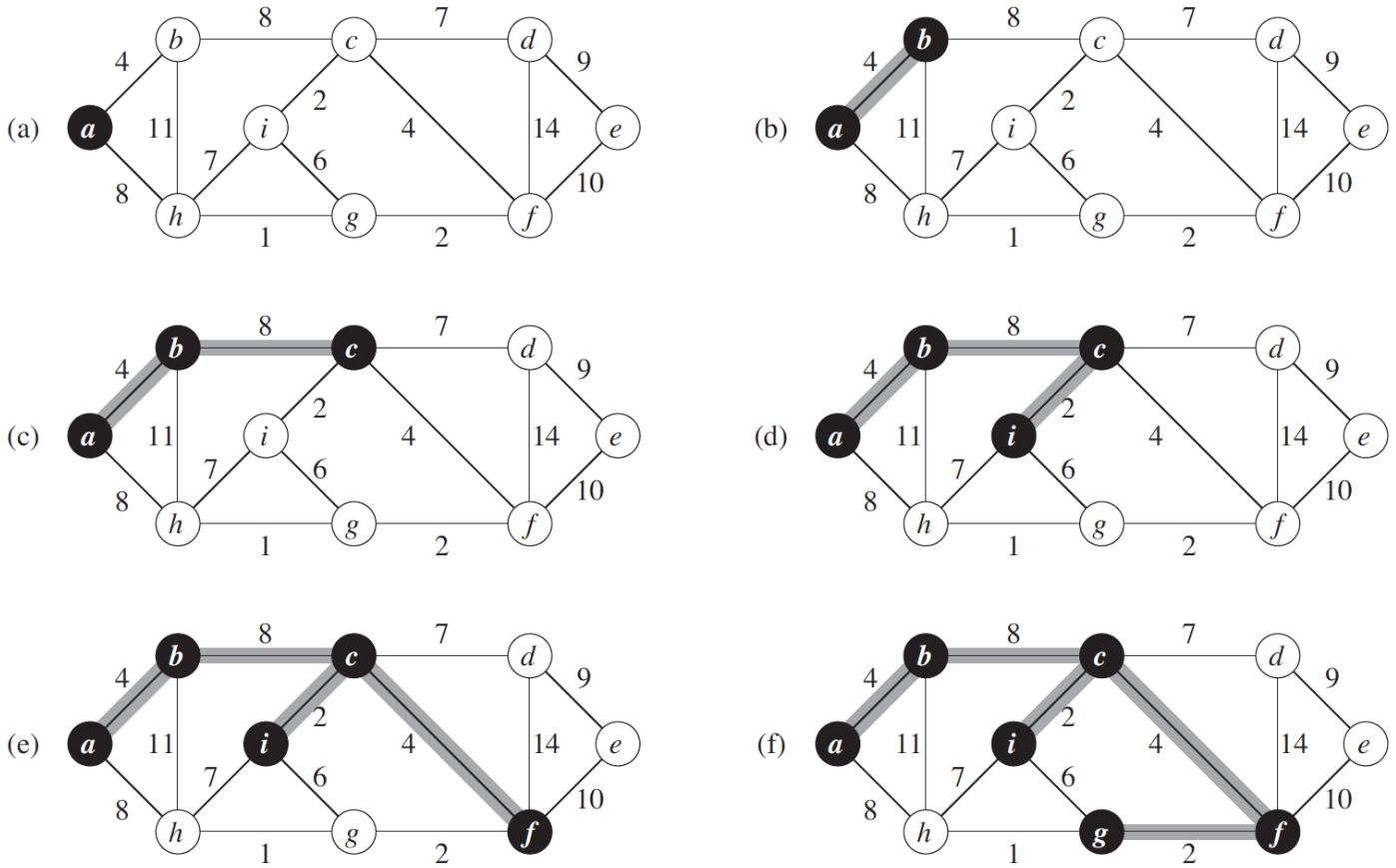

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- As with Dijkstra, faster implementation exists: $O(|V| \log |V| + |E|)$.

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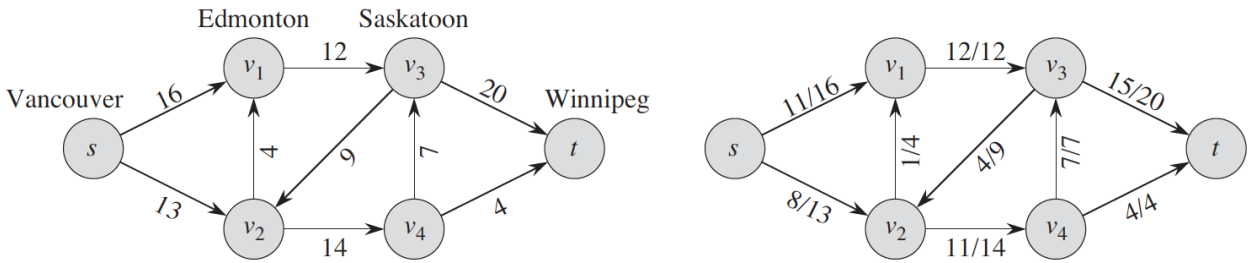
- We assume there are no self-loops (u, u).
- A *flow* from s to t is a function f where if (u, v) is an edge, then

$$f(u, v) \leq c(u, v)$$

and $f(u, v) = 0$ if there is no edge (u, v), and where for all u different from s and t,

$$\sum_v f(v, u) = \sum_v f(u, v).$$



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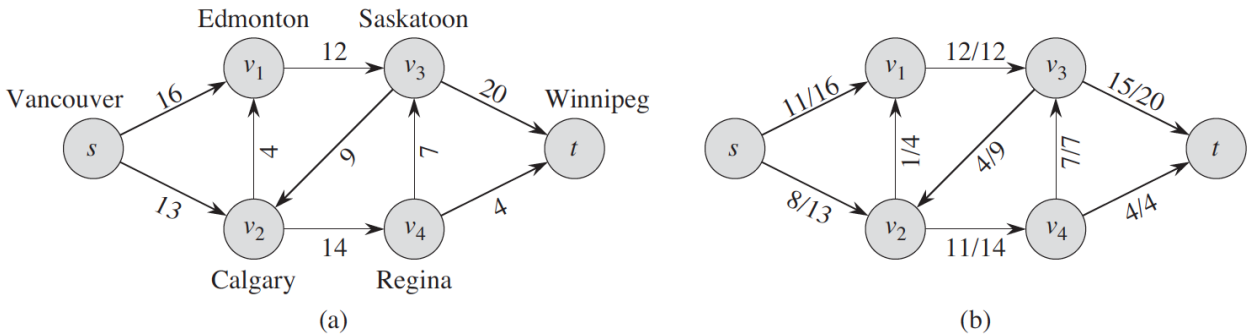

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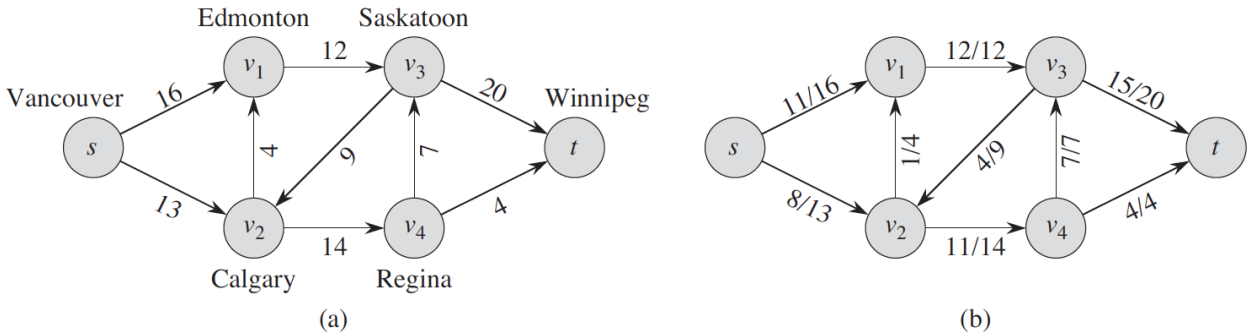

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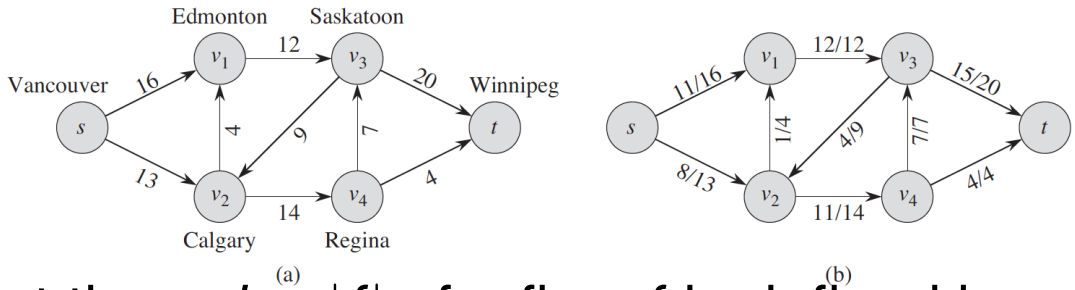

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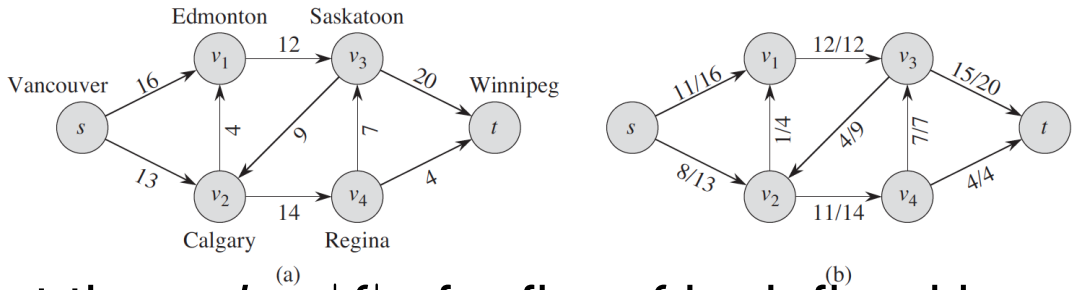

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- Why is it also true that

$$|f| = \sum_v f(v, t) - \sum_v f(t, v)?$$

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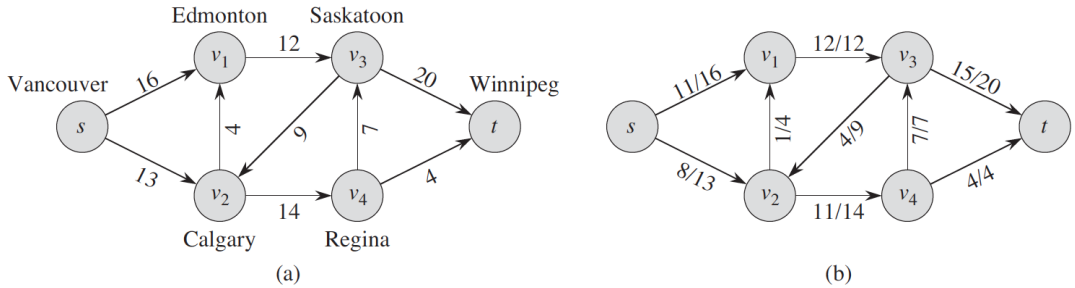

- We have

$$0 = \sum_u \left(\sum_v f(u,v) - \sum_v f(v,u) \right)$$

$$= \left(\sum_v f(s,v) - \sum_v f(v,s) \right) + \left(\sum_v f(t,v) - \sum_v f(v,t) \right)$$

$$+ \sum_{u \neq s,t} \left(\sum_v f(u,v) - \sum_v f(v,u) \right) =$$

$$= |f| + \left(\sum_v f(t,v) - \sum_v f(v,t) \right) + \sum_{u \neq s,t} 0,$$

so $\sum_v f(t,v) - \sum_v f(v,t) = -|f|$.

Next time!

Max-flow and min-cut