COMP 761: Lecture 28 – Binary Search Trees I

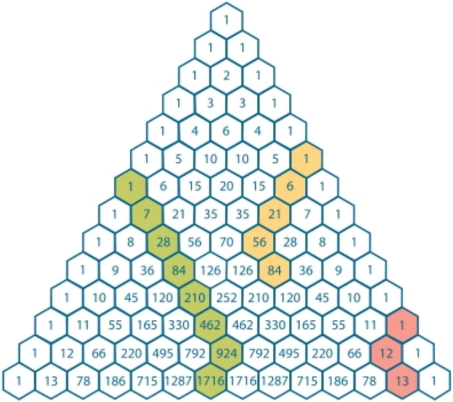
David Rolnick

November 9, 2020

Problem

Prove the Hockey Stick Identity:

$$\sum_{i=0}^{n-1} \binom{i+k}{k} = \binom{n+k}{k+1}.$$



Course Announcements

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- Problem Set 3 grades out, let Vincent and me know if you think something should be reconsidered.

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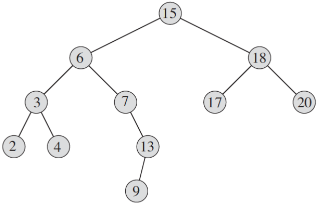

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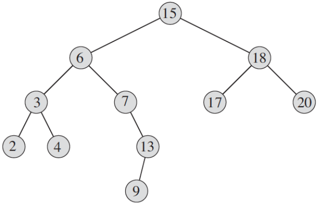

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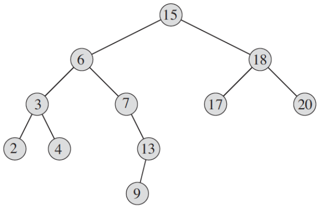

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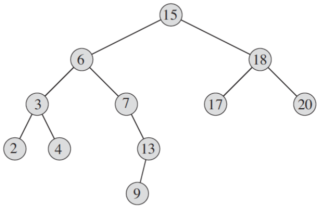

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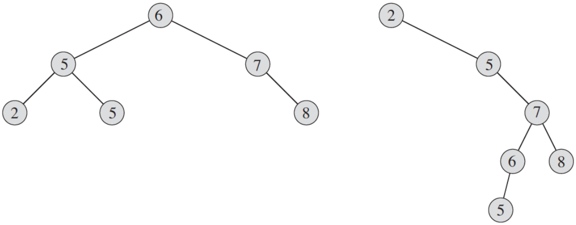
- Yes!



(a) (b)

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  - Insert and delete (add or remove a new key)

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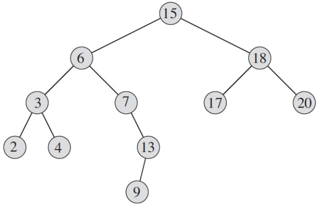

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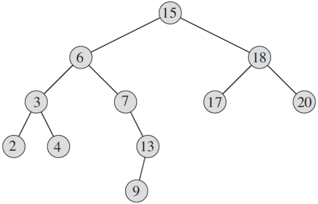

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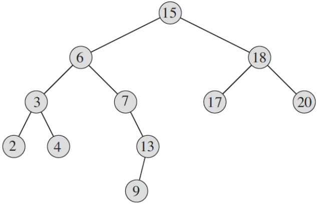

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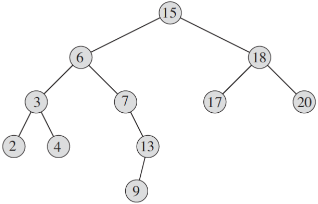

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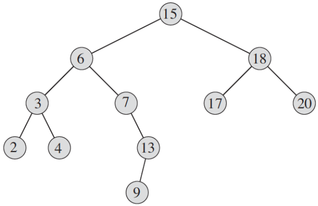

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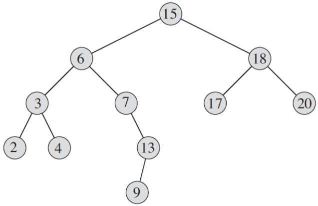

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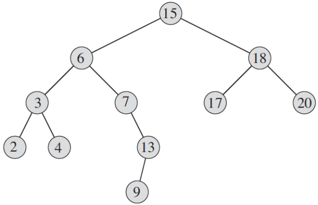

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- How long does this take?
- The time is $O(h)$, where h is the *height* of the tree (=maximum depth of all nodes).

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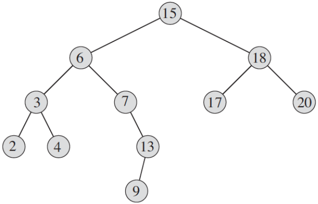

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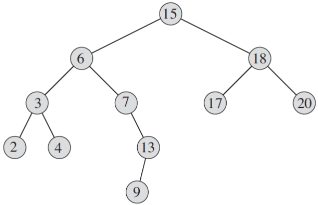

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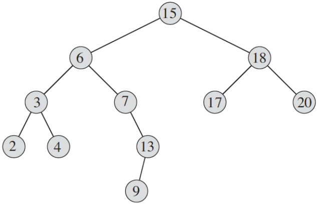

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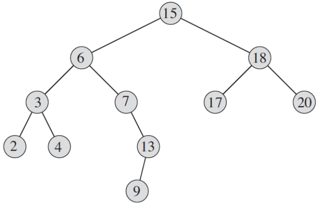

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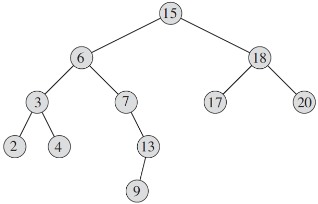

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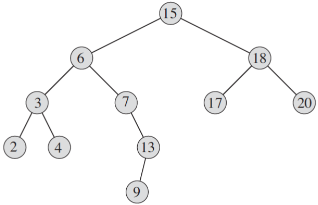

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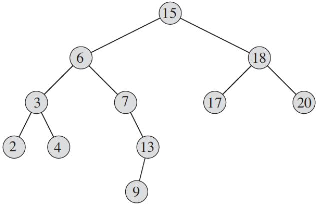

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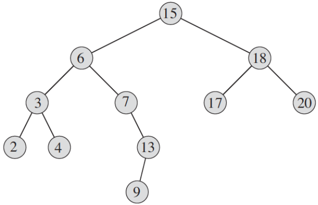

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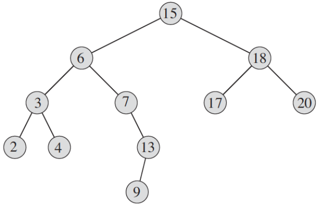

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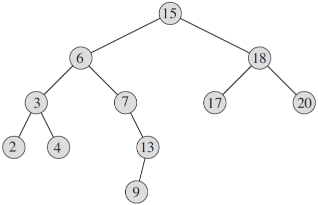

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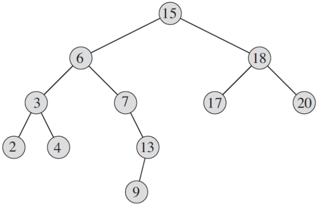

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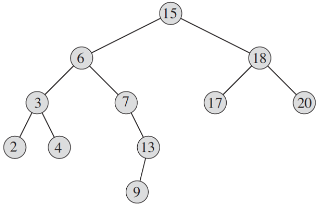

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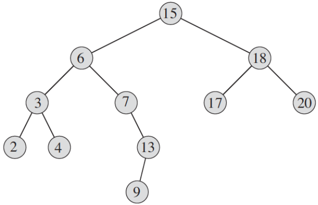

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- Unless x equals v or w.

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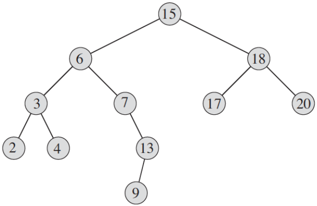

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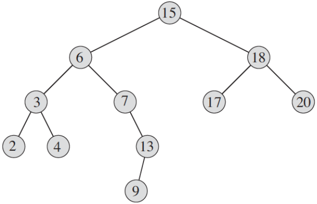

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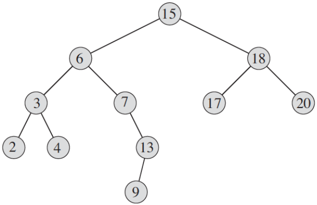

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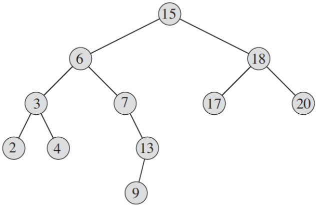

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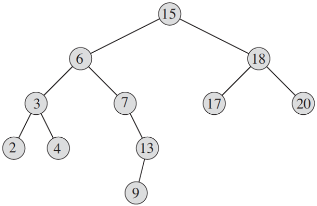

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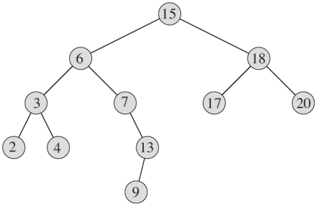

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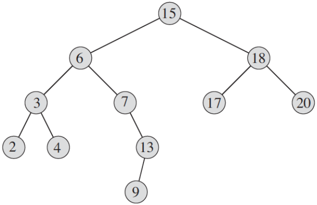

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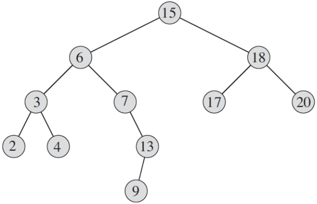

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- How do we know which case we are in?

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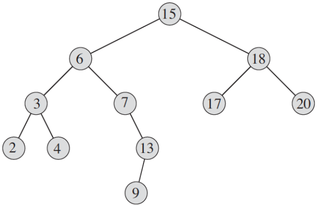

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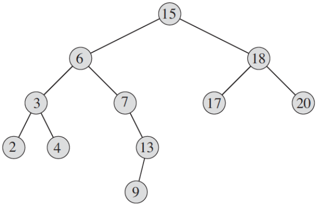
Successor and predecessor

Given a node v with key k_v, how to find the *successor* w to v, i.e. with the smallest k_w that is larger than k_v? (Let's assume here all keys are different.)



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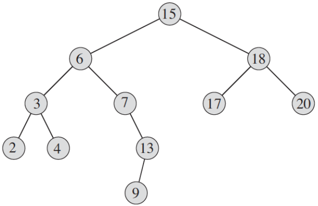

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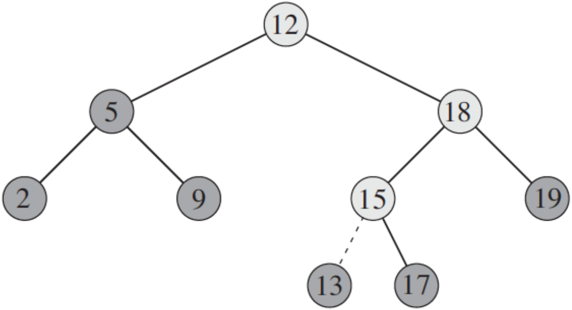

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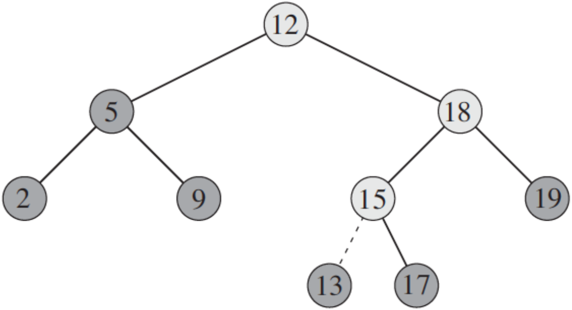

- This takes $O(h)$ time again.

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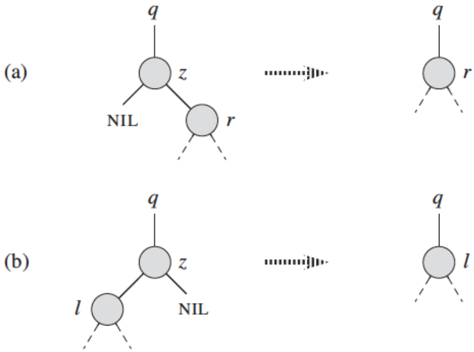
- Similarly if z is missing a right child.



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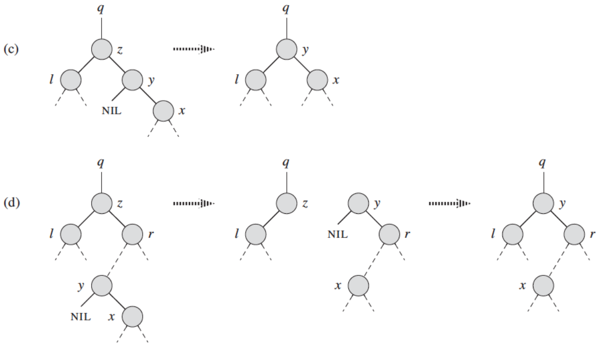
- Here is how we can do that:



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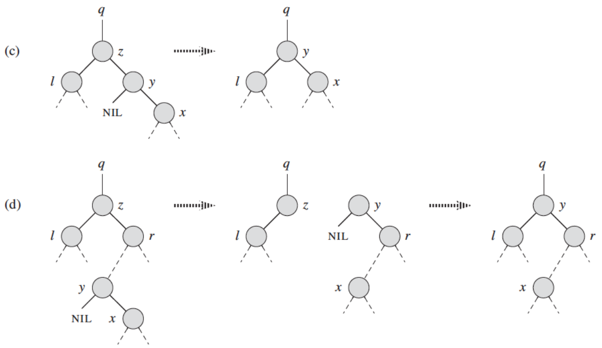

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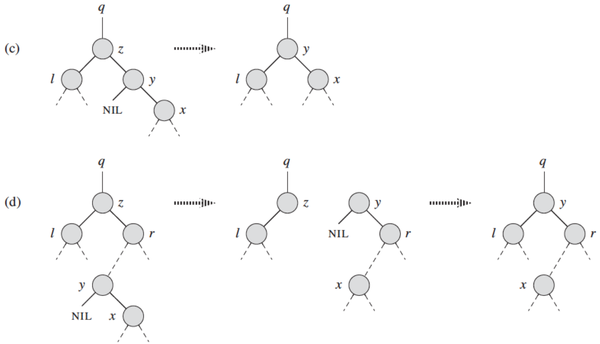

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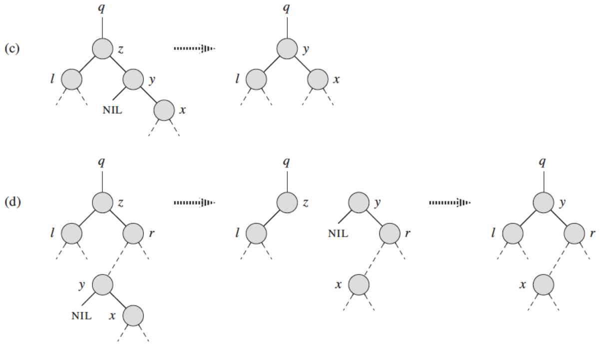

- How long does this process take?
- As before, we might have to go all the way down the tree, so $O(h)$.

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- Let's consider a *typical* binary search tree.
- Suppose that we insert $\{1, 2, \ldots, n\}$ into a binary search tree in random order. What is the expected height?

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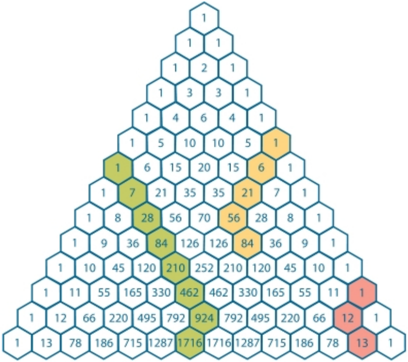

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- How can we go from m to $m+1$?

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- Where the last step is by Pascal's identity.

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- This is essentially the same as the weighted form of Jensen's inequality we have already seen:

$$\sum_{i=1}^{n} p_i f(x_i) \geq f\left(\sum_{i=1}^{n} p_i x_i\right)$$

if p_i are nonnegative with $\sum_{i=1}^{n} p_i = 1$.