COMP 761: Lecture 30 – Hashing

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November 13, 2020

Problem

Suppose we have integers $k_1 \neq k_2$, a prime number p, and integers a, b with a not divisible by p. Define the remainders mod p:

$$\begin{aligned} r_1 &= (ak_1 + b) \mod p \\ r_2 &= (ak_2 + b) \mod p. \end{aligned}$$

Prove that $r_1 \neq r_2$, and that it is possible to solve for a and b modulo p given p, r_1, r_2, k_1, k_2.

(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)

Course Announcements

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- Problem 3 – you can assume that you know the max flow already. (If you didn't know the max flow, the algorithm might take longer than $|E|$ steps.)

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- Similarly, deletion is a pain.

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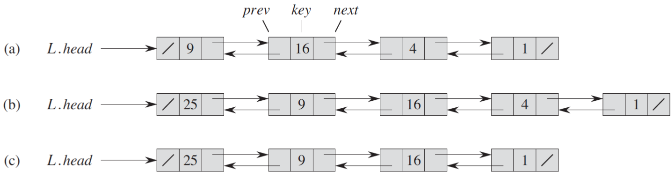

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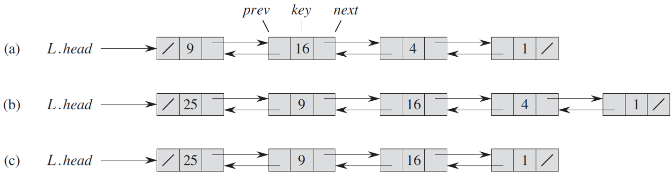

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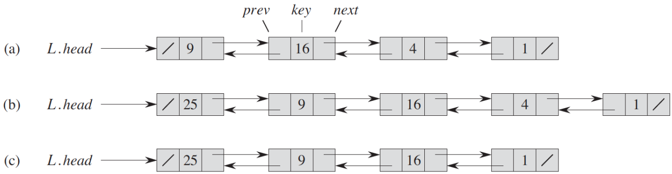

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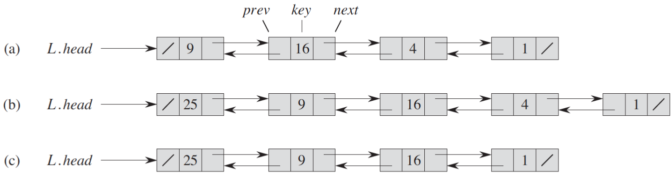

- So it is possible from any element to access the next one and previous one (*sequential access*), but not to jump immediately to any element.
- Getting the mth element takes $O(m)$ time.
- But insertion and deletion are $O(1)$, just insert and rearrange the pointers from the next and previous elements.

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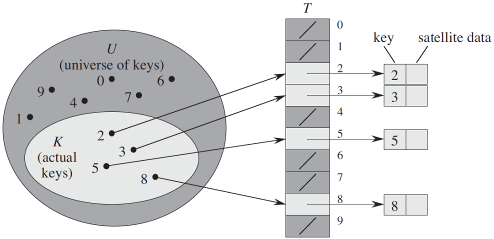

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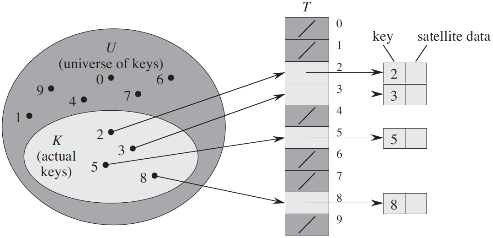

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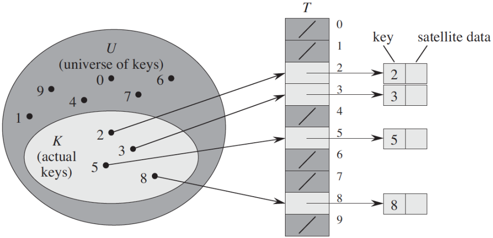

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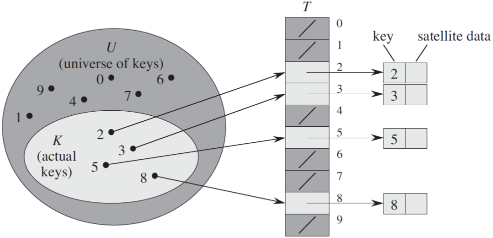

- What's a potential problem with this?
- There might be a lot of possible usernames.
- Would need a huge amount of memory and most of the slots would be unused.

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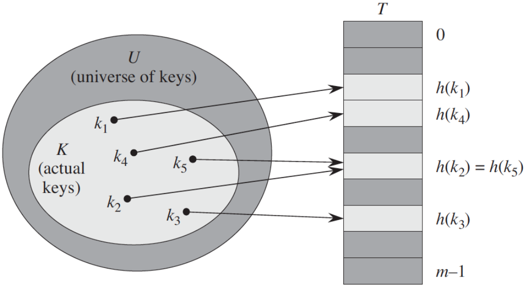

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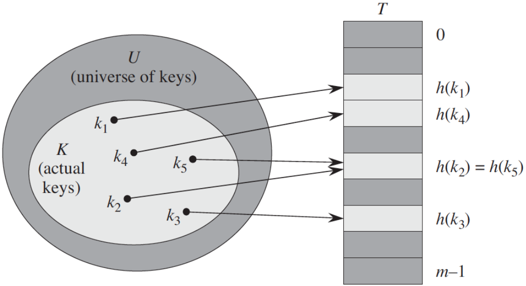

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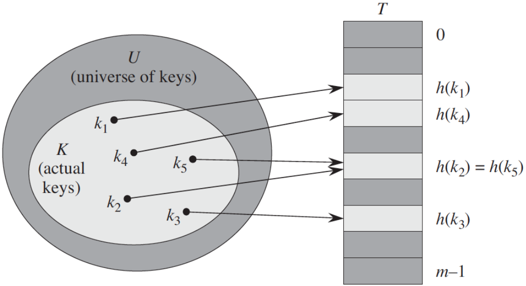

- What's a potential problem with this?
- We could have k_1 and k_2 where $h(k_1) = h(k_2)$ – which gets stored there?
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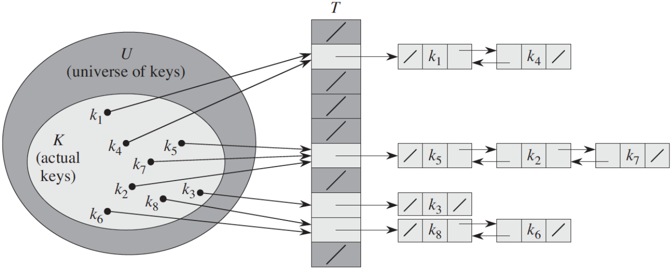
- First thing inserted there (key k_1) initializes the list.
- If $h(k_2) = h(k_1)$, then add a new entry for k_2 to the start of the list.
- This is called *chaining* within the hash table.



- Hopefully none of the lists gets very long, since finding something within a linked list is slow.

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- Let's break into two cases, depending on whether it's in the table already or not.

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where we write $1+\alpha$ since we don't know if $\alpha = n/m$ is bigger or less than 1.

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- k might be any of these with equal probability.
- Let X_{ij} be the indicator variable that $h(k_i) = h(k_j)$.
- What is the expected value of the position of k_i within its slot?
- The expected position is 1 plus the expected number of elements inserted after we inserted k_i:

$$1 + \mathbb{E}\left[\sum_{j=i+1}^{n} X_{ij}\right].$$

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- Since k could be any k_i with equal probability, the expected position of k in its slot is:

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&= 1 + \frac{1}{nm}((n-1) + (n-2) + \ldots + 1)
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- Since $n/m = \alpha$, this is $\Theta(1+\alpha)$, so the total time to search for k (including $\Theta(1)$ for computing $h(k)$) is $\Theta(1) + \Theta(1+\alpha) = \Theta(1+\alpha)$.

Simple hashing methods

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- In practice, we would want to avoid certain m, in particular powers of two, since often residues mod 2^s have patterns for certain data.
- A prime number m is often good.

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- We say that $\mathcal{H}$ is *universal* if for any two different keys k_1, k_2, the probability of picking h with $h(k_1) = h(k_2)$ is at most $1/m$.

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- We will prove that this set of hash functions is universal.