COMP 761: Lecture 32 – Miscellaneous

David Rolnick

November 18, 2020

Problem

Find a situation where Ford-Fulkerson can take either 2 steps or 2 million steps depending on how you pick the augmenting paths.

(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)

Course Announcements

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- Office hours: Vincent at **11 am** Thurs (different time this week), David at 10 am Fri.

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- Alternatively, they can be computed as they are needed, but then stored in case they must be reused. This is called *memoization* (not a typo – we are making *memos* of useful solutions).

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- Alternatively, they can be computed as they are needed, but then stored in case they must be reused. This is called *memoization* (not a typo – we are making *memos* of useful solutions).
- Why is this called "dynamic"? Richard Bellman chose this word because:

> *It also has a very interesting property as an adjective, and that is it's impossible to use the word dynamic in a pejorative sense. ... Thus, I thought dynamic programming was a good name. It was something not even a Congressman could object to.*

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- If we multiply $(AB)C$, then we have $(10)(100)(5) = 5000$ for the 10×5 matrix AB and $(10)(5)(50) = 2500$ operations for the 10×50 matrix $(AB)C$, for a total of

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- If we multiply $A(BC)$, then we have $(100)(5)(50) = 25000$ for the 100×50 matrix BC and $(10)(100)(50) = 50000$ for the 10×50 matrix $A(BC)$, for a total of

$$25000 + 50000 = 75000.$$

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- We could just try all possible ways to do it, e.g.

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- Roughly how big is P_n?

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- Grow exponentially, so trying all of these possibilities is a bad idea.

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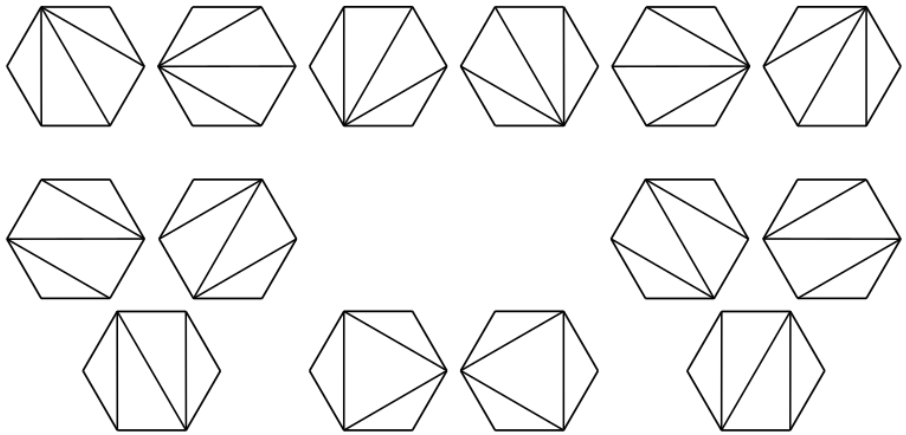

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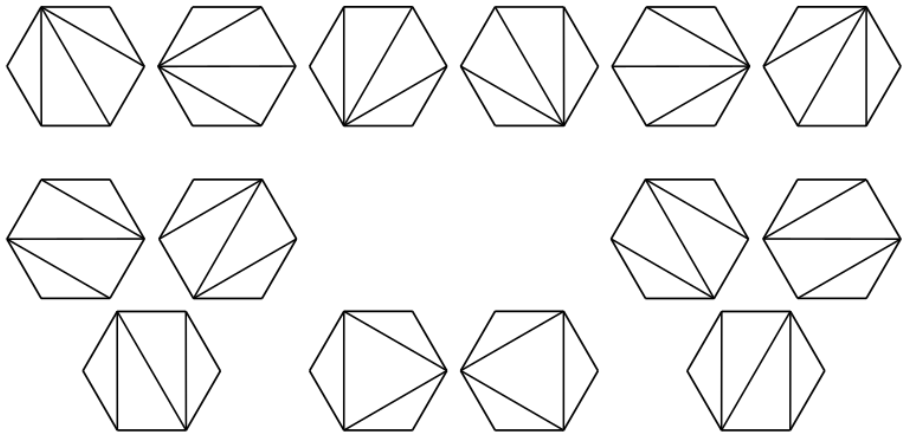

- Closed form expression:

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

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- We have

$$f(A_1, \ldots, A_n) = \min_k \left(f(A_1, \ldots, A_k) + p_1 p_{k+1} p_{n+1} + f(A_{k+1}, \ldots, A_n) \right).$$

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- So we can precompute the solutions to these subproblems.

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- What is a good base case?

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$$\sum_{i=1}^{n} \sum_{j=i}^{n} \Theta(j - i) = \sum_{i=1}^{n} \sum_{j=0}^{n-i} \Theta(j) = \sum_{i=1}^{n} \Theta((n - i)^2) = \Theta(n^3).$$

where we use $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ from the start of the course.

Shortest paths

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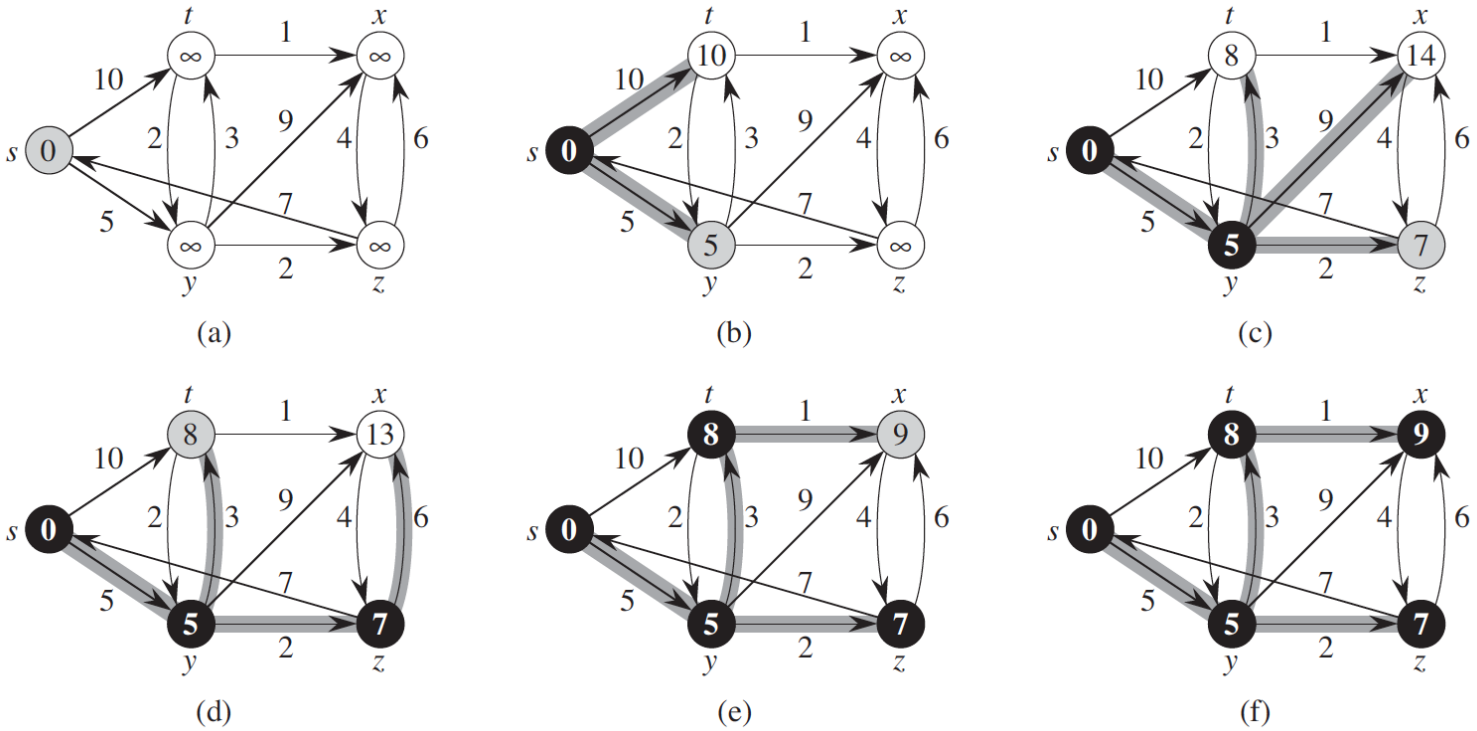

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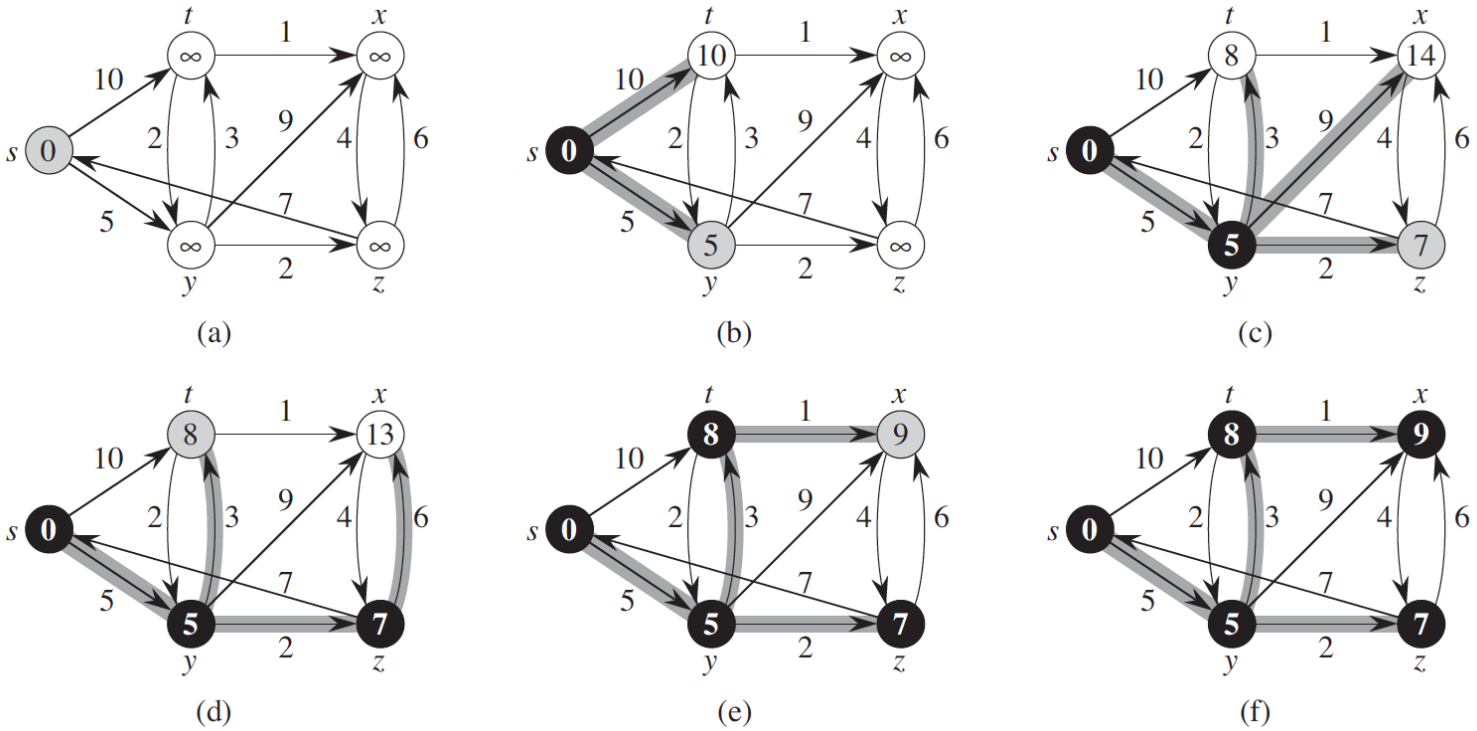

- Key property: If P is a shortest path between u and v and w is an intermediate vertex on that path, we can break P at w into path P_1 and P_2 where P_1 is a shortest path from u to w and P_2 is a shortest path from w to v.

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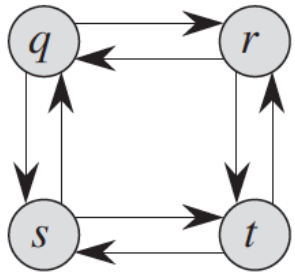

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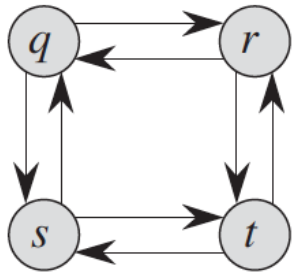

- In the graph above, (q, r, t) is a longest path from q to t, but if we break it into (q, r) and (r, t), neither of these is a longest path ((q, s, t, r) and (r, q, s, t) are longer, respectively).

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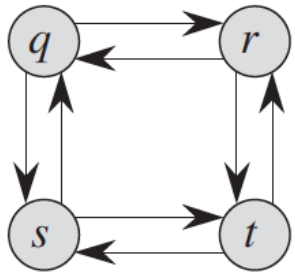

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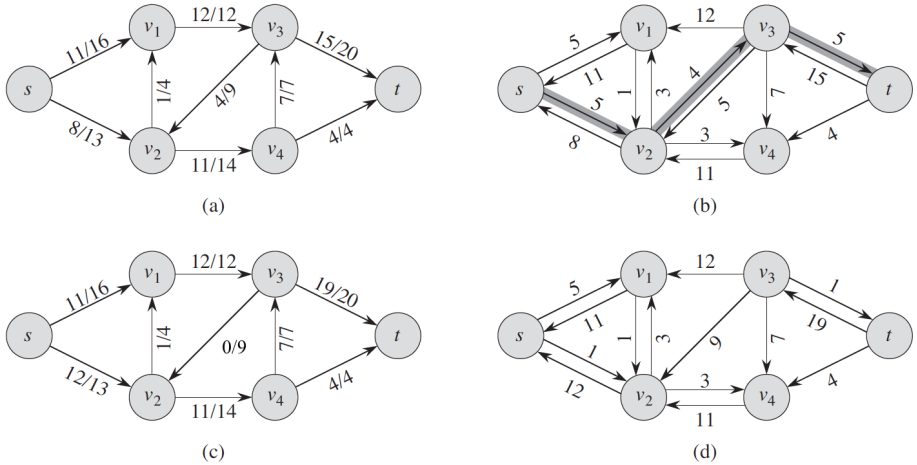
- Dynamic programming only works if we can dissect a solution into the solutions of smaller problems.

Context for Problem 3 – augmenting paths in max flow



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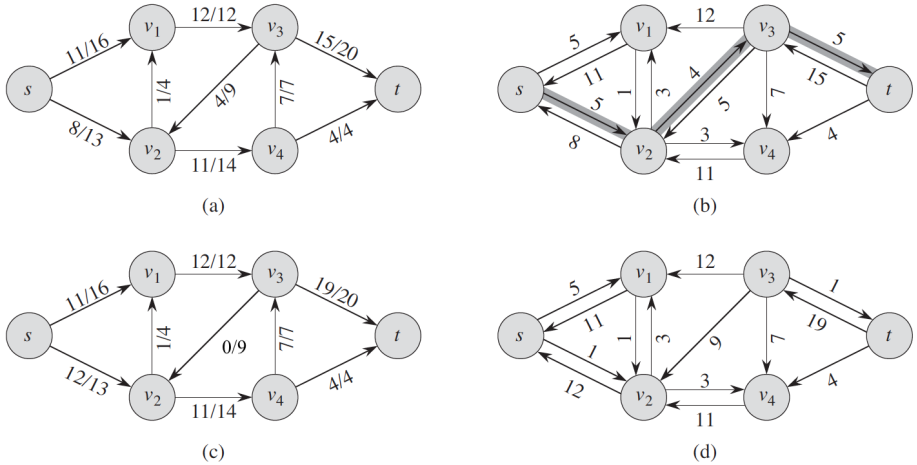

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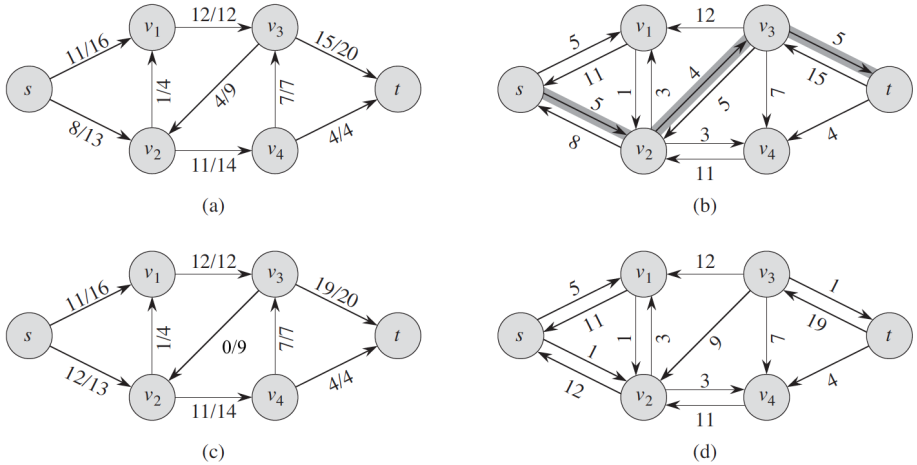

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- If (u, v) is an edge in G, then both (u, v) and (v, u) are edges in G_f, with *residual capacities*

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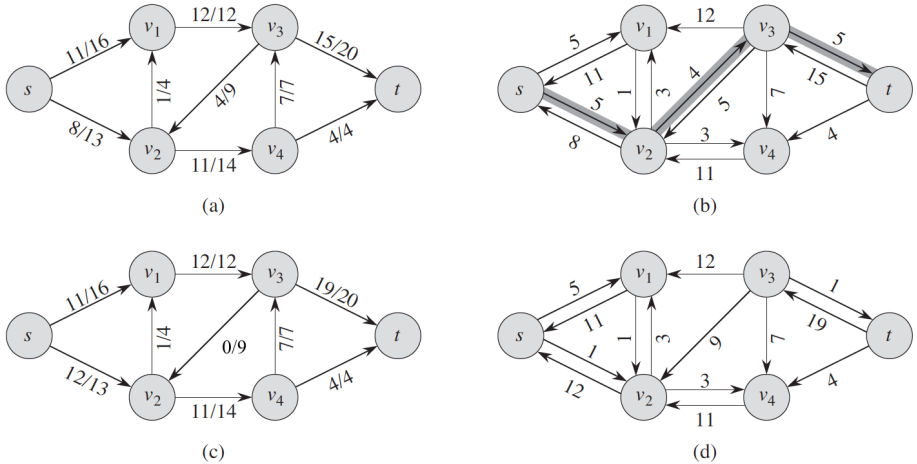

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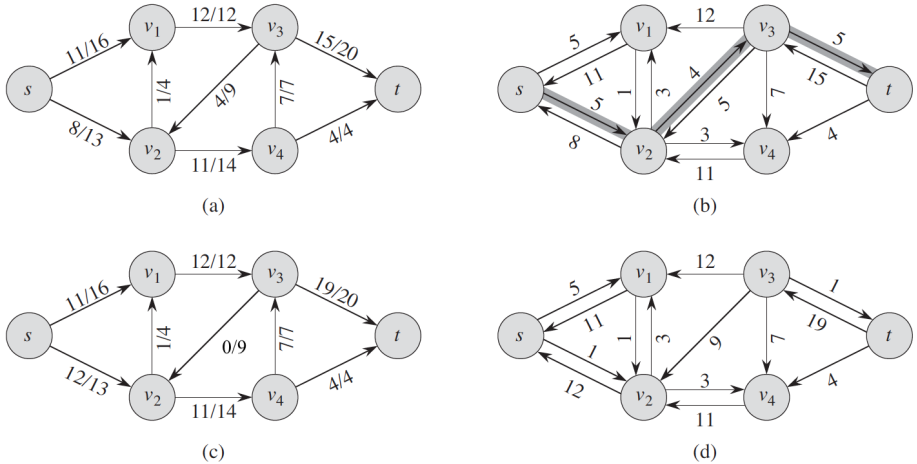

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- Each augmenting path gives a flow with the min capacity of all edges in the path. This flow can be added to the flow f to increase $|f|$.

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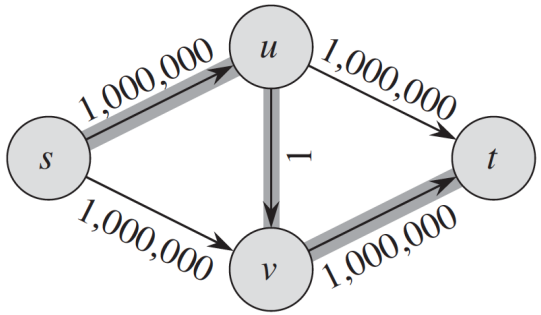

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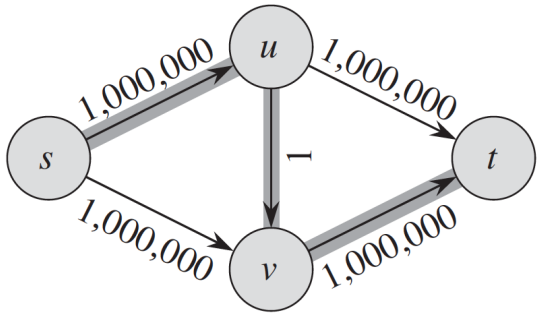

- We could also pick this bad sequence of augmenting paths, and take 2 million steps: