# COMP 761: Lecture 36 – Neural Networks I

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November 27, 2020

# Problem

Suppose you are given datapoints $x^1, x^2, \ldots, x^n \in \mathbb{R}^d$ and binary labels $y^1, y^2, \ldots, y^n \in \{+1, -1\}$ such that the data is *linearly separable*, that is, for some weight vector $w$:

$$y^k = \text{sign}(w \cdot x^k) = \text{sign}\left(\sum_{i=1}^{d} w_i x_i^k\right), \text{ for every } k.$$

Describe an algorithm to find such a weight vector $w$.

*(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)*

# Course Announcements

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- On the problem set, totally fine to take as a given that the problem of determining whether a Hamiltonian cycle exists in a graph in NP-complete.

# Fun sidenote (not about neural nets)

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- This came up in office hours today.

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- 2-SAT is like 3-SAT except the clauses in the formula have 2 variables each, not 3, e.g.:

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- Is 2-SAT NP-complete?

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- Case 2 ($x_2$ is TRUE) is similar.

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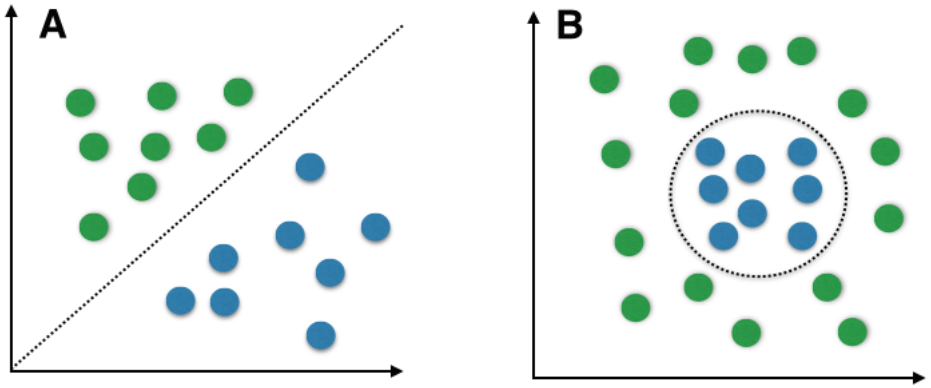

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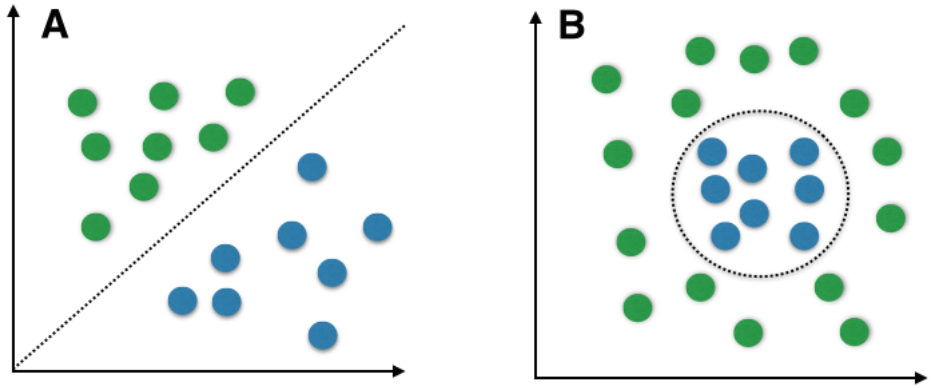


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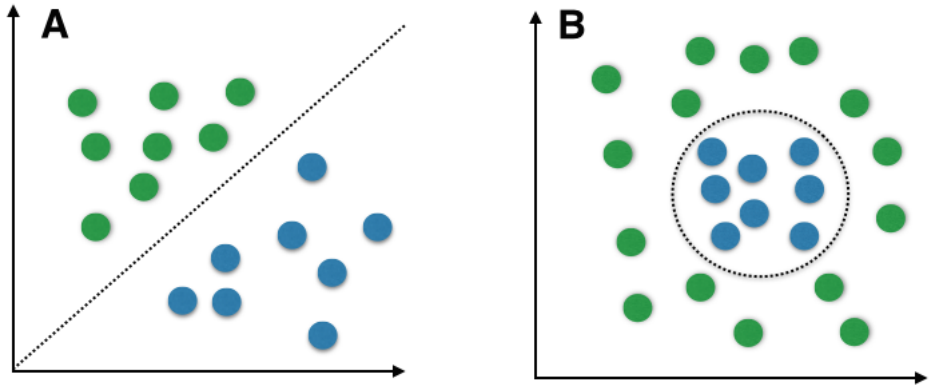


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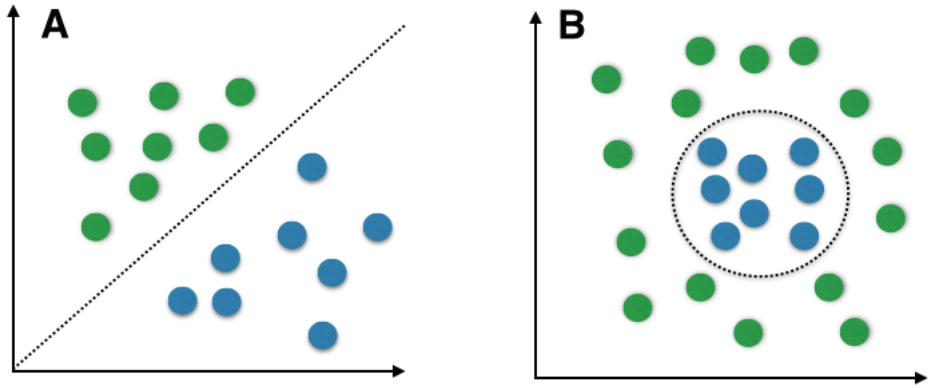


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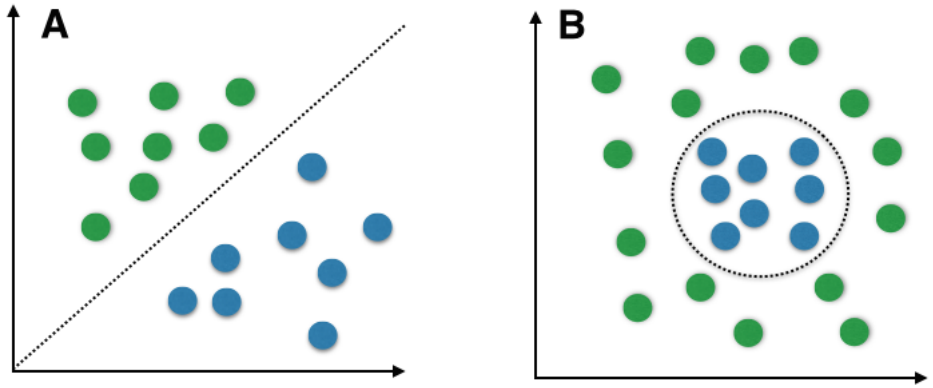


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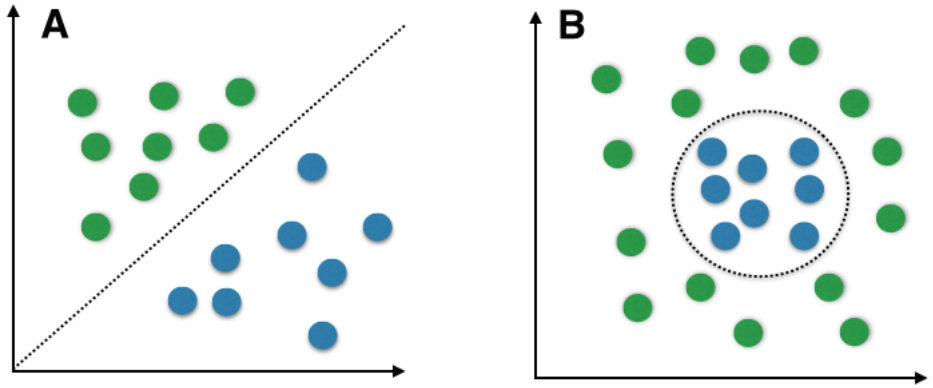


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- Given labels $y^1, y^2, \ldots, y^n \in S$, where $y^k = y(x^k)$.
- Need to come up with a function $f(x) \approx y(x)$ to predict new labels.

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- In some binary classification tasks, the data will be *linearly separable*, i.e. the labels $y(x) \in \{+1, -1\}$ satisfy:

$$y(x) = \text{sign}(w \cdot x) = \text{sign}\left(\sum_{i=1}^{d} w_i x_i\right),$$

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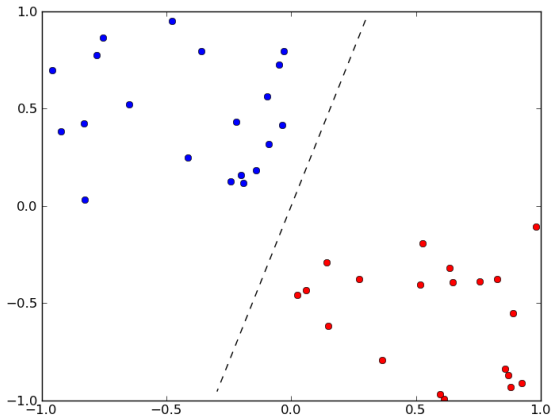

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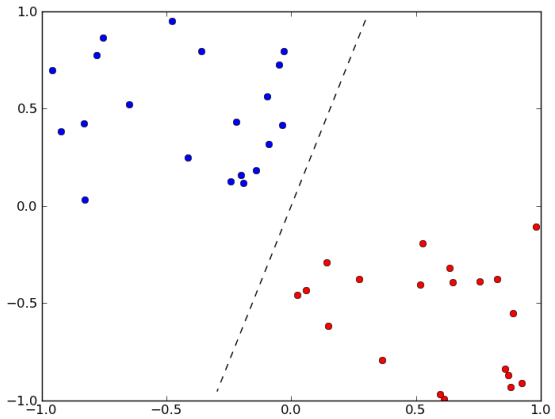


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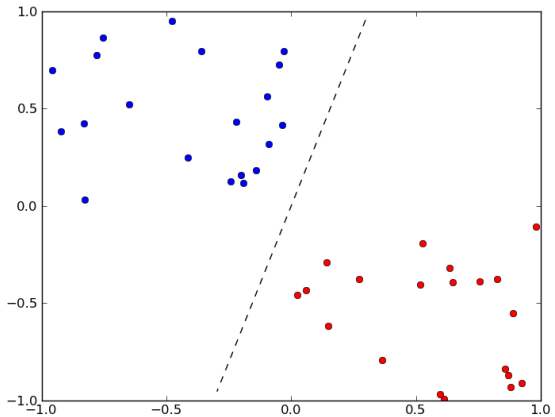


- We learn a function $w \cdot x$ from observed data $x^1, x^2, \ldots, x^n$ and labels $y^1, y^2, \ldots, y^n \in \{+1, -1\}$.
- Then we can predict the class of a new data point $x$.

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  - Find some $x^k$ such that $y^k \neq \text{sign}(w^j \cdot x^k)$.
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- Let's see why this works.

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- The data is linearly separable – i.e., there is a unit weight vector $w^*$ and a *margin* $\gamma > 0$ such that for every $k$, we have $w^* \cdot x^k > \gamma$ if $y^k = +1$ and $w^* \cdot x^k < -\gamma$ if $y^k = -1$. We can summarize this as $y^k(w^* \cdot x^k) > \gamma$.

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- Since $w^*$ is a unit vector, we have

$$j\gamma < w^* \cdot w^{j+1} \leq ||w^*|| \cdot ||w^{j+1}|| = ||w^{j+1}||, \text{ proving the claim.}$$

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- Note: This algorithm only works if the data is exactly linearly separable.