COMP 761: Lecture 38 – Neural Networks III

David Rolnick

December 2, 2020

Problem

Describe a gradient descent approach to learning the weights of a perceptron.

(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)

Course Announcements

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- Last class tomorrow (Thursday) at 4 pm Montreal.

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- Office hours as usual Thurs 10:30 am (Vincent) and Fri 10 am (David).

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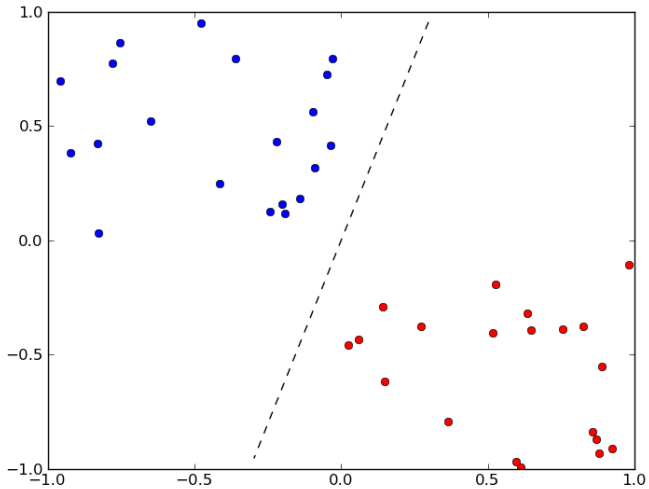

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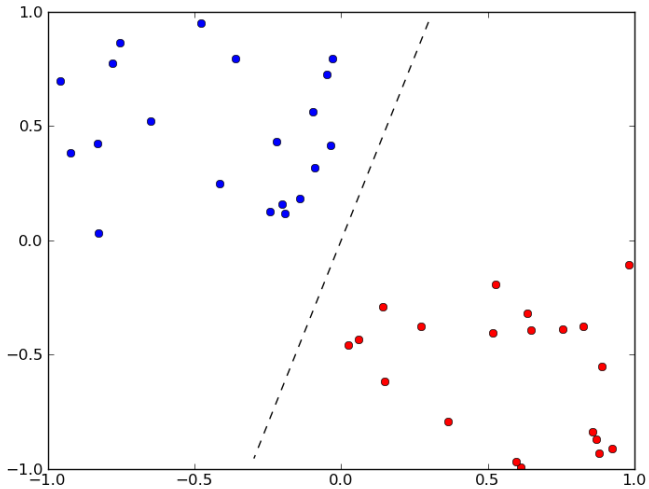

- What if the points are not linearly separable?

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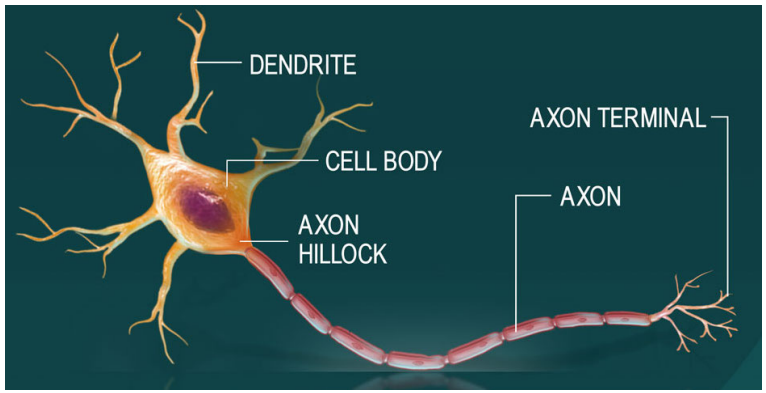

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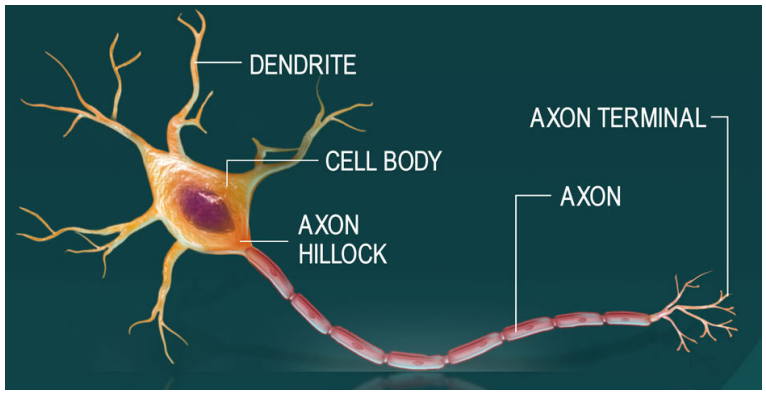

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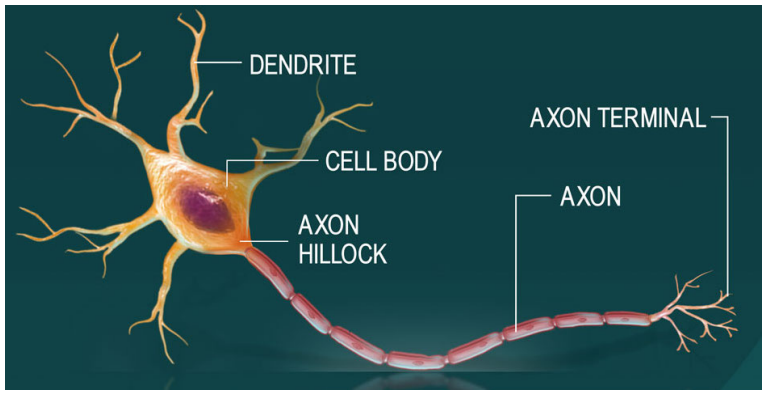

- Whole brain = many neurons that fire together in patterns that in turn lead to other firing patterns.

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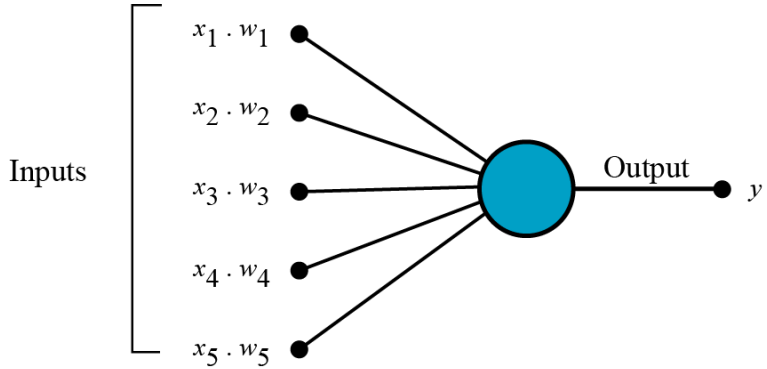

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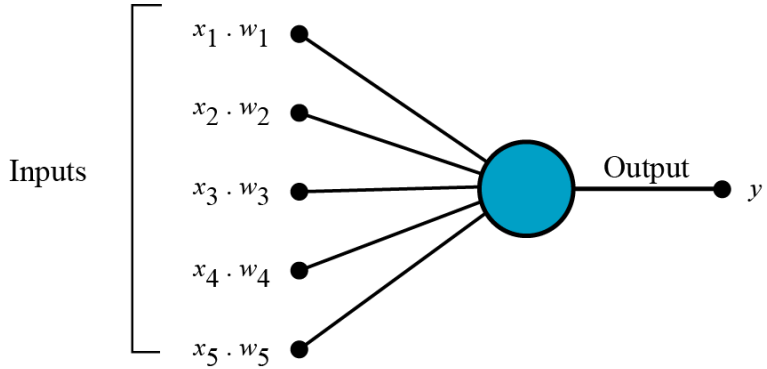

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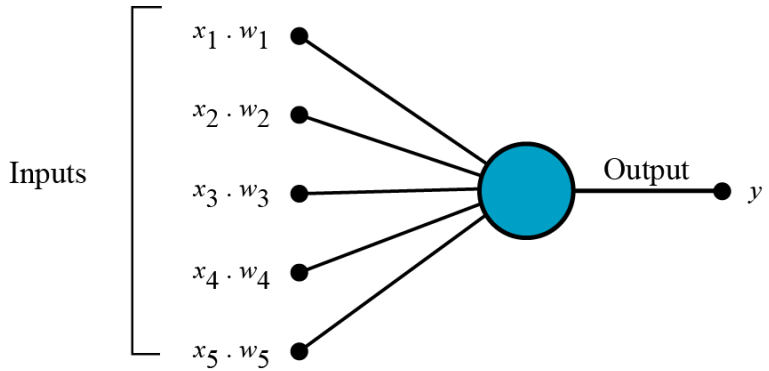

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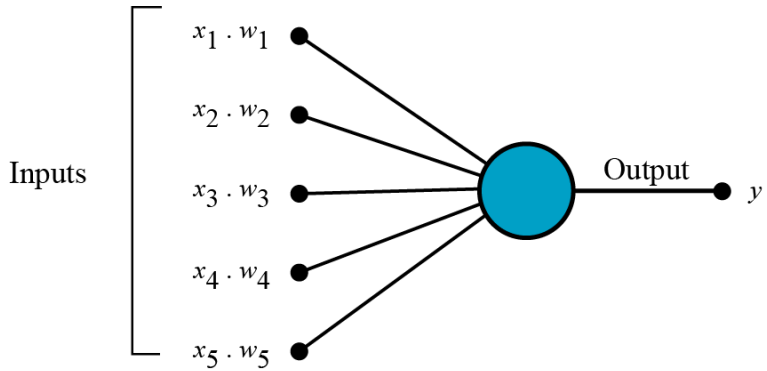

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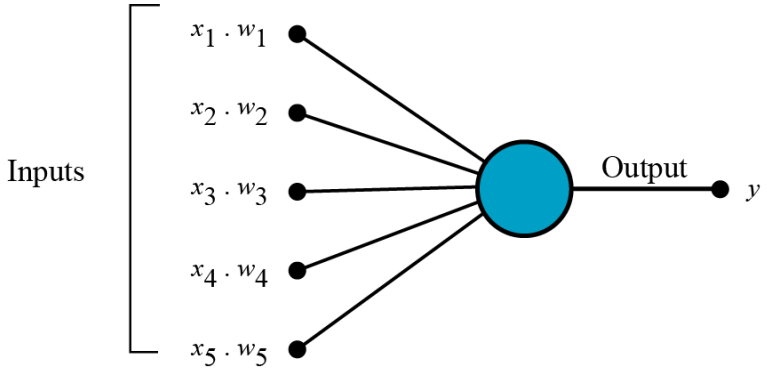

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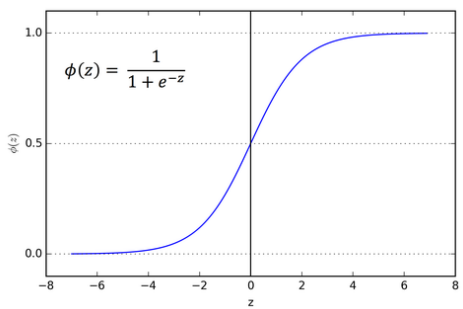
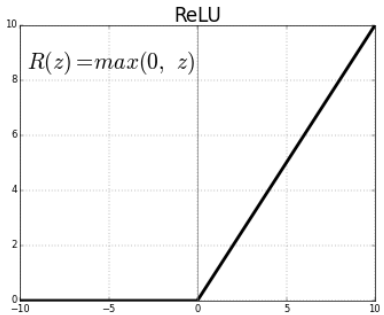
- Can generalize to $y(x) = \sigma(w \cdot x - b)$, where σ is a nonlinear function (since neurons aren't quite as exact as on and off).
- Two common choices $\sigma(z)$ are sigmoid $1/(1 + e^{-z})$ and ReLU $\max(0, z)$:





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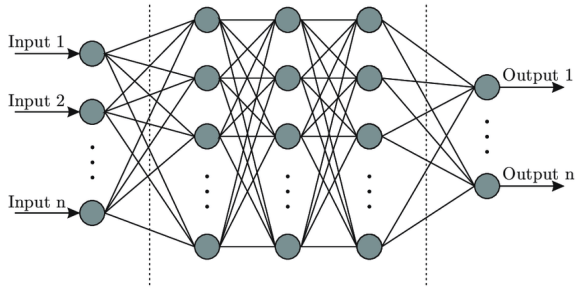

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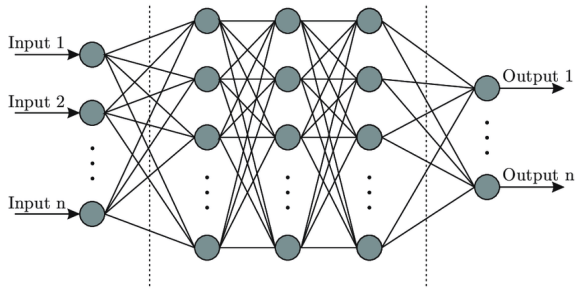

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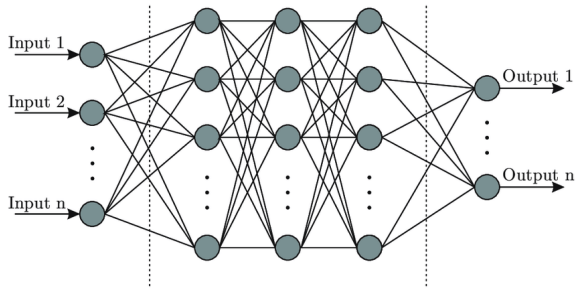

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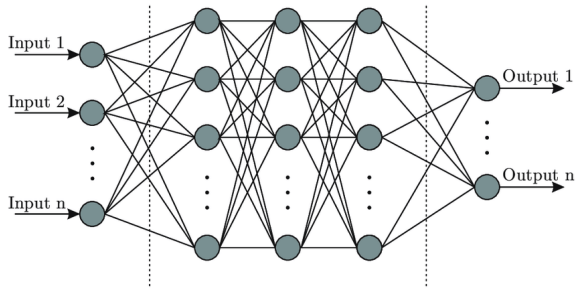

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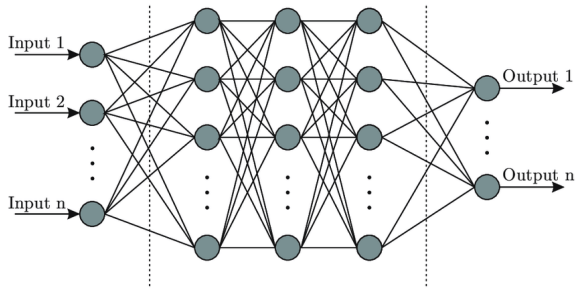

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- So each layer is a vector of perceptrons applied to the previous layer.

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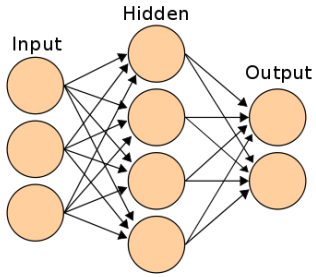

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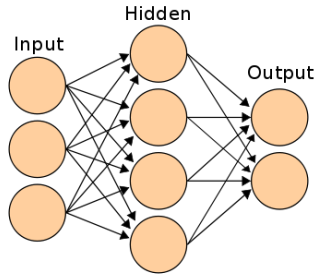

- Let's compute the output of this multilayer perceptron at the point $x = [1, 0, 0]$ with

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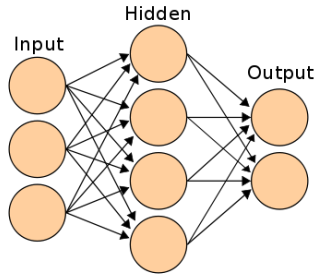

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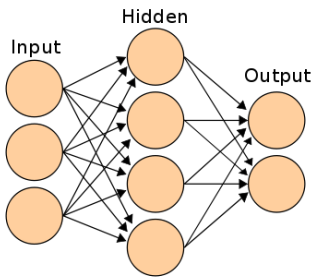

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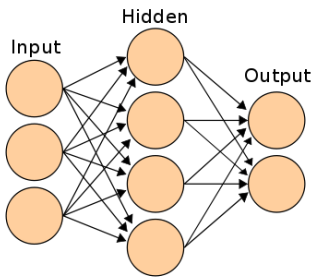

- We change the weight matrices W^1 and W^2 and the bias vectors b^1 and b^2 to create a function that best replicates the patterns observed in data.

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- For example, in our network with 1 hidden layer, increasing b^2 will increase $f_\theta(x^i)$ for every i.

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&= 2(\sigma(w \cdot x^i - b) - y^i) \times \sigma'(w \cdot x^i - b) \times x^i_j.
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- For a general neural network, we can similarly use the chain rule to compute the gradients with respect to each parameter of the network:

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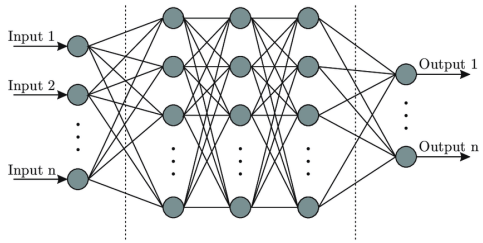

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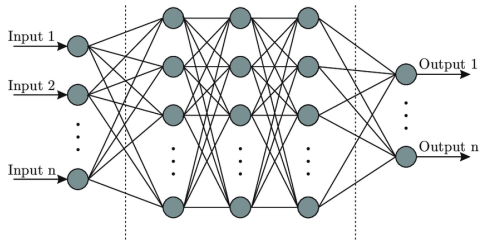

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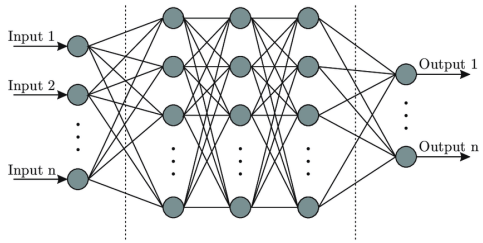

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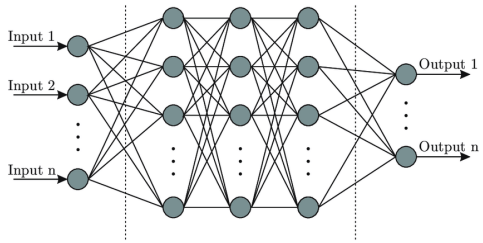

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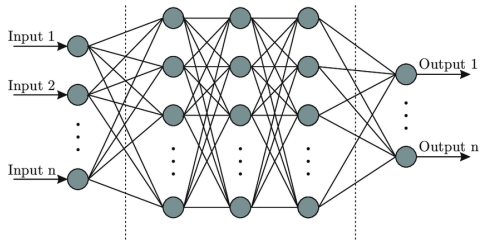

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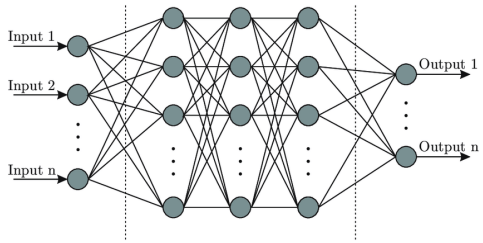

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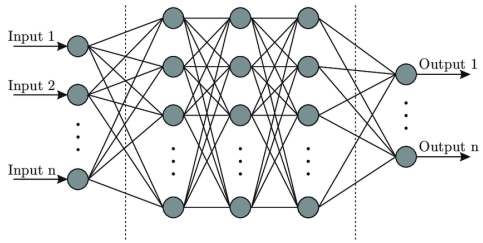

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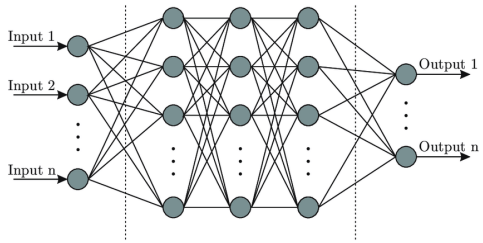

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- (Even for a perceptron, gradient descent can be better than the perceptron learning rule if the data is not perfectly linearly separable.)

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- However, we still don't really know why deep learning works so well!

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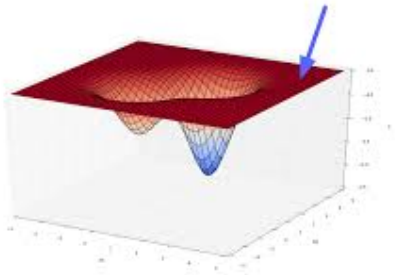

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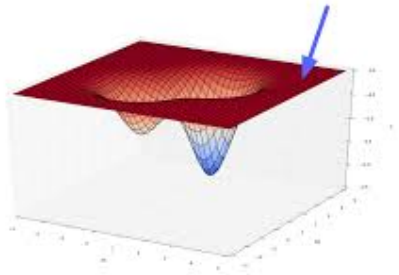
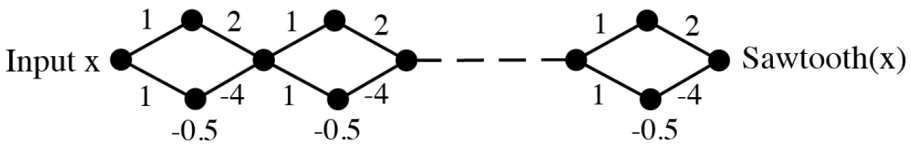
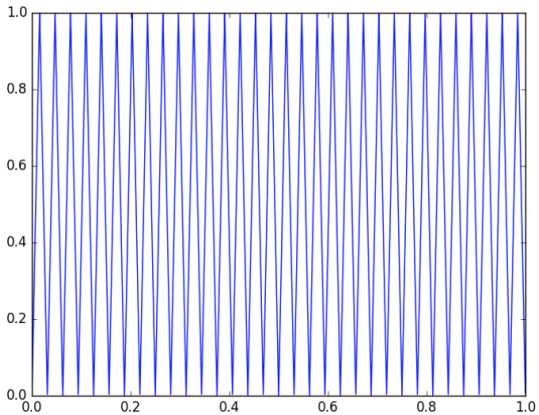


- For example, this neural network expresses the sawtooth function:





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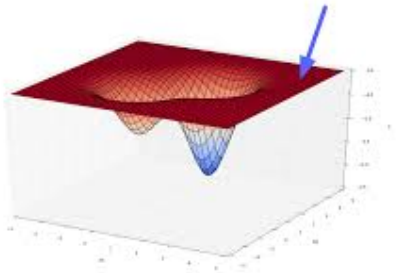

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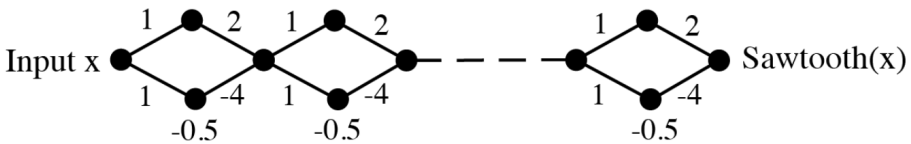

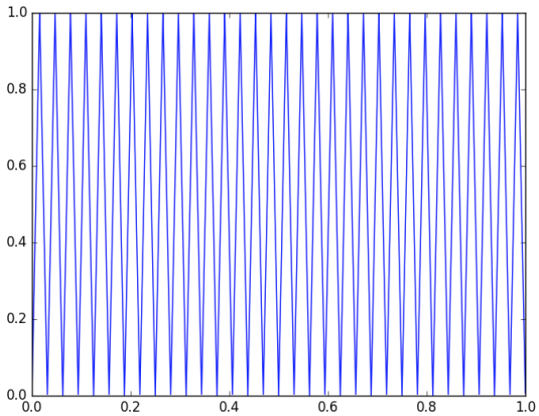

- If you were to set the weights perfectly, it would work, but unless you are close already the gradient is very low.

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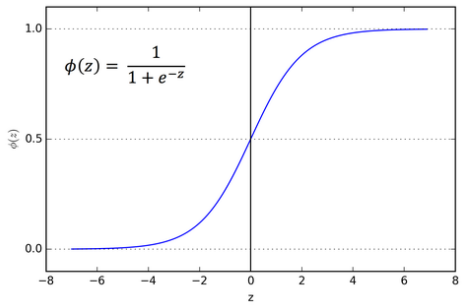

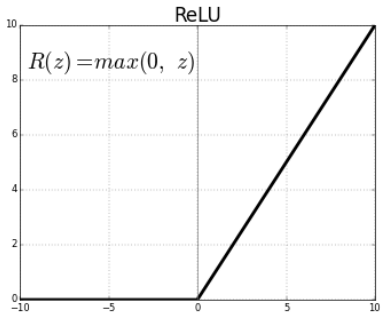

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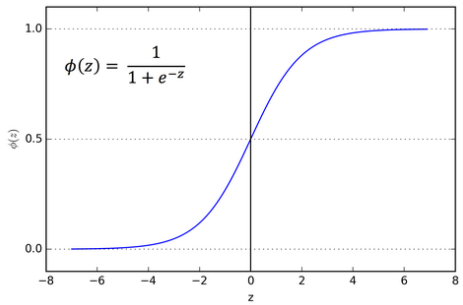

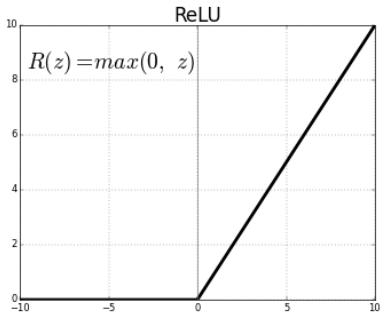

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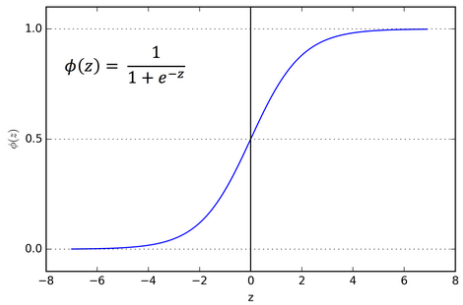

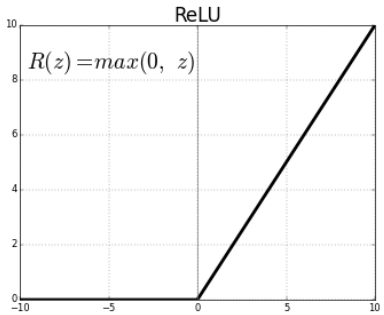

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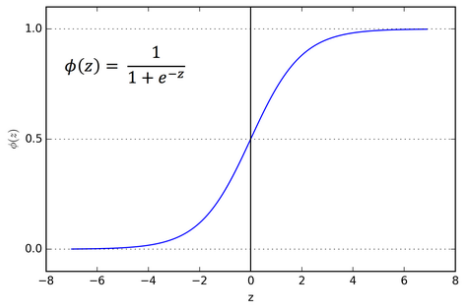

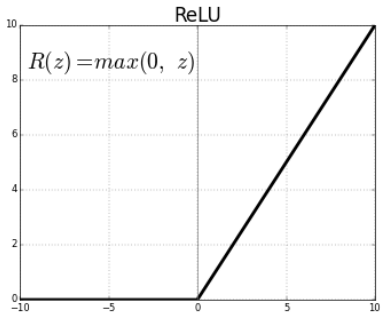

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- Typically each ReLU will be 0 for some inputs and > 0 for others, which means about half of the weights to be learning at a particular datapoint.