COMP 761: Lecture 39 – Final Class

David Rolnick

December 3, 2020

Problem

Describe three dice A, B, C with numbers on their sides, such that if we roll them all, A is likely to be higher than B, B is likely to be higher than C, and C is likely to be higher than A.

(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)

Course Announcements

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- Office hours at normal time (10 am Montreal) Friday.

Unsolved problems

Goldbach's conjecture

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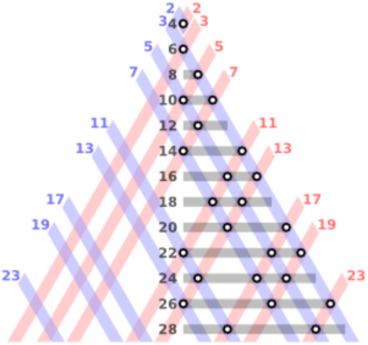
- Has been checked up to 10^{18}.



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- Paul Erdős: "Mathematics may not be ready for such problems."
- Has been tested up to 2^{68} (about 300 quintillion).

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- Currently, the best known algorithm is $O(n^{2.3728596})$.
- We don't know if this is the best.

Graph isomorphism problem

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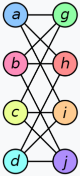
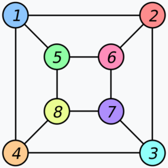
- Two graphs are said to be *isomorphic* if we can reorder the vertices so they line up.

Graph G	Graph H	An isomorphism between G and H
		$f(a) = 1$ $f(b) = 6$ $f(c) = 8$ $f(d) = 3$ $f(g) = 5$ $f(h) = 2$ $f(i) = 4$ $f(j) = 7$

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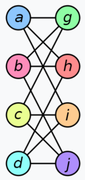	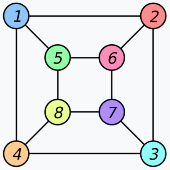	$f(a) = 1$ $f(b) = 6$ $f(c) = 8$ $f(d) = 3$ $f(g) = 5$ $f(h) = 2$ $f(i) = 4$ $f(j) = 7$

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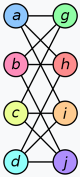	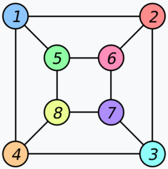	$f(a) = 1$ $f(b) = 6$ $f(c) = 8$ $f(d) = 3$ $f(g) = 5$ $f(h) = 2$ $f(i) = 4$ $f(j) = 7$

- Let ISOMORPHISM(G, H) be the problem of working out whether two graphs G and H are isomorphic.
- May be in P, may be NP-complete, or neither (not known).

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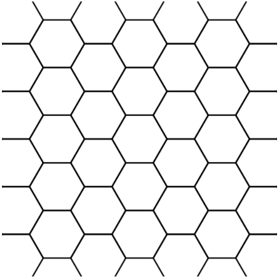

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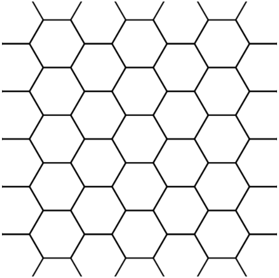

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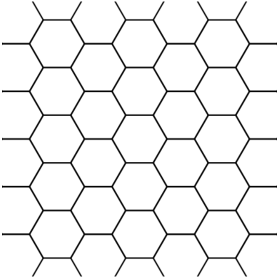

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- This one was actually proven in 1999!

Surprising math / paradoxes

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- But since they overlap, the colors must be the same! So all the horses in S are the same color. Done with the induction.

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- What is wrong with this argument??
- S_1 and S_2 only overlap if $n > 1$, so even though the base case of $n = 1$ works, we can't go from 1 to 2.

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- Wait, now medicine A is performing better? What happened?
- The reason this happened was because for A, the tests happened mostly in cats (where both medicines perform well), while for B, the tests happened mostly in dogs (where neither medicine performs well).
- This is one reason why it is important to break out confounding variables in studies.

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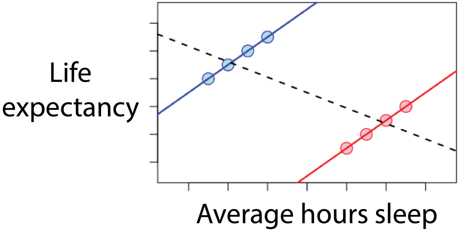
- This is an (imaginary) plot of amount of sleep vs life expectancy:



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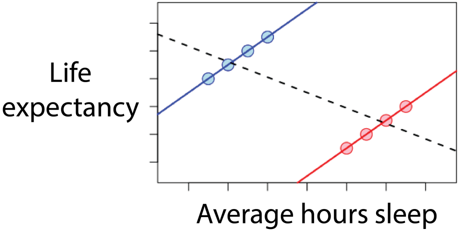

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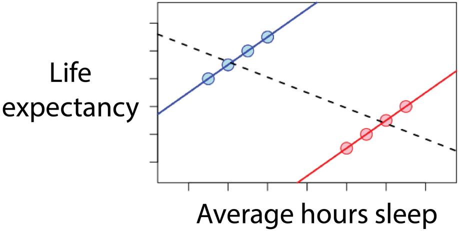

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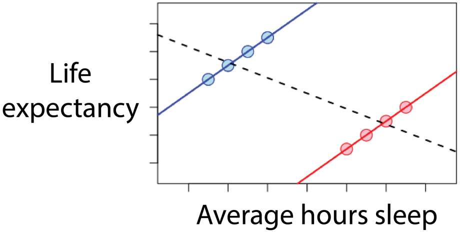

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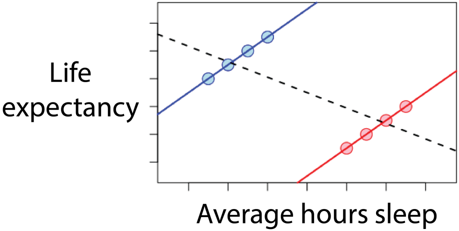

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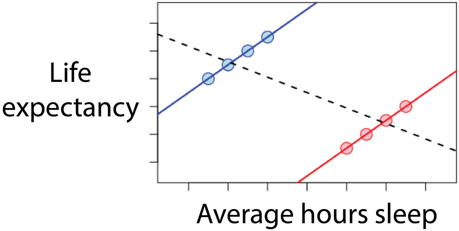

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- For each group individually, more sleep is good.

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  - A: sides 2, 2, 4, 4, 9, 9.

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  - B: sides 1, 1, 6, 6, 8, 8.

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  - A: sides 2, 2, 4, 4, 9, 9.
  - B: sides 1, 1, 6, 6, 8, 8.
  - C: sides 3, 3, 5, 5, 7, 7.

Non-transitive dice

- Suppose we have 3 dice:
  - A: sides 2, 2, 4, 4, 9, 9.
  - B: sides 1, 1, 6, 6, 8, 8.
  - C: sides 3, 3, 5, 5, 7, 7.
- What is the probability that A rolls a higher number than B?

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- And likewise $5/9$ probability for $C > A$.

Player 1 chooses die **A**
Player 2 chooses die **C**

C \ A	**2**	**4**	**9**
3	C	A	A
5	C	C	A
7	C	C	A

Player 1 chooses die **B**
Player 2 chooses die **A**

A \ B	**1**	**6**	**8**
2	A	B	B
4	A	B	B
9	A	A	A

Player 1 chooses die **C**
Player 2 chooses die **B**

B \ C	**3**	**5**	**7**
1	C	C	C
6	B	B	C
8	B	B	B

Non-transitive dice

- Suppose we have 3 dice:
  - A: sides 2, 2, 4, 4, 9, 9.
  - B: sides 1, 1, 6, 6, 8, 8.
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Player 2 chooses die **C**

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3	C	A	A
5	C	C	A
7	C	C	A

Player 1 chooses die **B**
Player 2 chooses die **A**

A \ B	1	6	8
2	A	B	B
4	A	B	B
9	A	A	A

Player 1 chooses die **C**
Player 2 chooses die **B**

B \ C	3	5	7
1	C	C	C
6	B	B	C
8	B	B	B

- So each of the dice is likely to beat the next one in the list!

Non-transitive dice

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  - A: sides 2, 2, 4, 4, 9, 9.
  - B: sides 1, 1, 6, 6, 8, 8.
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Player 2 chooses die **C**

C \ A	2	4	9
3	C	A	A
5	C	C	A
7	C	C	A

Player 1 chooses die **B**
Player 2 chooses die **A**

A \ B	1	6	8
2	A	B	B
4	A	B	B
9	A	A	A

Player 1 chooses die **C**
Player 2 chooses die **B**

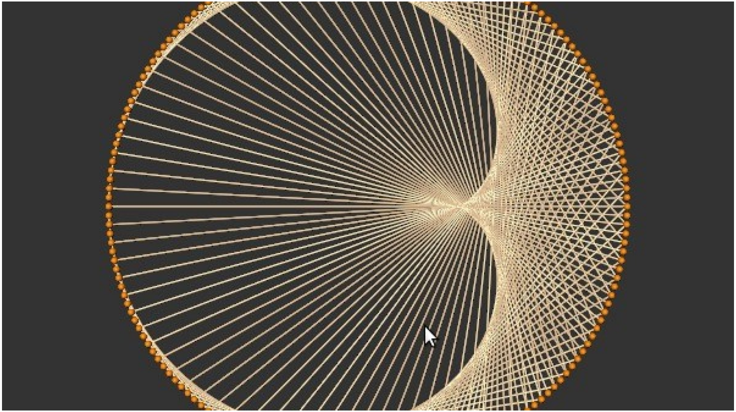
B \ C	3	5	7
1	C	C	C
6	B	B	C
8	B	B	B

- So each of the dice is likely to beat the next one in the list!
- Not impossible, just counter-intuitive.

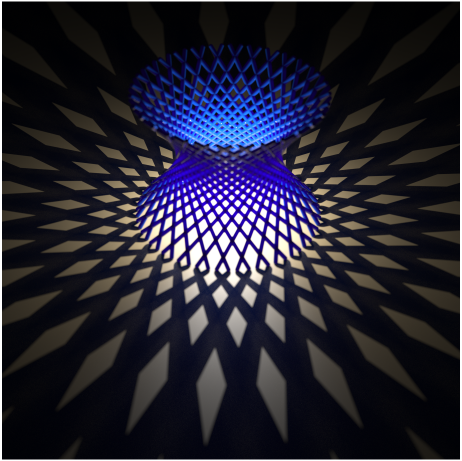
Mathematical art

Based on dodecahedron

Cardioid



One-sheet hyperboloid



Tessellations

Penrose tiling